

# Primitive axial algebras of Jordan type

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Version of: 7 March 2014

## Abstract

An *axial algebra* over the field  $\mathbb{F}$  is a commutative algebra generated by idempotents whose adjoint action has multiplicity-free minimal polynomial. For semisimple associative algebras this leads to sums of copies of  $\mathbb{F}$ . Here we consider the first nonassociative case, where adjoint minimal polynomials divide  $(x-1)x(x-\eta)$  for fixed  $0 \neq \eta \neq 1$ .

Jordan algebras arise when  $\eta = \frac{1}{2}$ , but our motivating examples are certain Griess algebras of vertex operator algebras and the related Majorana algebras. We study a class of algebras, including these, for which axial automorphisms like those defined by Miyamoto exist, and there classify the 2-generated examples. For  $\eta \neq \frac{1}{2}$  this implies that the Miyamoto involutions are 3-transpositions, leading to a classification.

## 1 Introduction

Throughout we consider commutative  $\mathbb{F}$ -algebras  $A$  where  $\mathbb{F}$  is a field of characteristic not equal to two. We emphasize that our algebras will usually be nonassociative and may not have an identity element.

For the element  $a$  of  $A$  and  $\lambda \in \mathbb{F}$ , the  $\lambda$ -eigenspace for the adjoint  $\mathbb{F}$ -endomorphism  $\text{ad}_a$  of  $A$  will be denoted  $A_\lambda(a)$  (where we allow the possibility  $A_\lambda(a) = 0$ ). If  $A$  is an associative algebra and  $a$  is an idempotent element, then  $A = A_1(a) \oplus A_0(a)$ —the adjoint of the idempotent is semisimple with at most the two eigenvalues 0 and 1. Here we are interested in the minimal nonassociative case—semisimple idempotents whose adjoint eigenvalues are drawn from the set  $\Lambda = \{1, 0, \eta\}$  for some  $\eta \in \mathbb{F}$  with  $0 \neq \eta \neq 1$ .

An idempotent whose adjoint is semisimple will be called an *axis*. A commutative algebra generated by axes is then an *axial algebra*. The commutative algebra  $A$  over  $\mathbb{F}$  (not of characteristic two) is a *primitive axial algebra of Jordan type  $\eta$*  provided it is generated by a set of axes with each member  $a$  satisfying:

- (a)  $A = A_1(a) \oplus A_0(a) \oplus A_\eta(a)$ .
- (b)  $A_1(a) = \mathbb{F}a$ .
- (c)  $A_0(a)$  is a subalgebra of  $A$ .
- (d) For all  $\delta, \epsilon \in \pm$ ,

$$A_\delta(a)A_\epsilon(a) \subseteq A_{\delta\epsilon}(a),$$

where  $A_+(a) = A_1(a) \oplus A_0(a)$  and  $A_-(a) = A_\eta(a)$ .

Examples include Jordan algebras that are generated by idempotents [Ja68]. These occur for  $\eta = \frac{1}{2}$ , although this is the case in which we say the least. Instead our motivation comes from the values  $\eta = \frac{1}{4}$  and  $\eta = \frac{1}{32}$ , which arise as special cases of  $\Lambda = \{1, 0, \frac{1}{4}, \frac{1}{32}\}$ . Algebras of this latter type are provided by Griess algebras associated with vertex operator algebras and Majorana algebras [Iv09, Ma05, Mi96, Sa07].

A major accomplishment in the Griess algebra case was Sakuma's Theorem [Sa07] which classified all 2-generated subalgebras. See also [Iv09, IPSS10, HRS13]. The following similar theorem is a central result of this paper.

**(1.1). THEOREM.** *Let  $\mathbb{F}$  be a field of characteristic not two with  $\eta \in \mathbb{F}$  for  $0 \neq \eta \neq 1$ . Let  $A$  be a primitive axial  $\mathbb{F}$ -algebra of Jordan type  $\eta$  that is generated by two axes. Then we have one of the following:*

- (1)  *$A$  is an algebra  $\mathbb{F}$  of type 1A over  $\mathbb{F}$ ;*
- (2)  *$A$  is an algebra  $\mathbb{F} \oplus \mathbb{F}$  of type 2B over  $\mathbb{F}$ ;*
- (3)  *$A$  is an algebra of type  $3C(\eta)$  of dimension 3 over  $\mathbb{F}$ ;*
- (4)  *$\eta = -1$  and  $A$  is an algebra of type  $3C(-1)^*$  of dimension 2 over  $\mathbb{F}$ ;*
- (5)  *$\eta = \frac{1}{2}$  and  $A$  is isomorphic to the 3-dimensional symmetric Jordan Clifford algebra  $\text{Cl}^J(\mathbb{F}^2, b_\delta)$ , where the symmetric bilinear form  $b_\delta$  on  $\mathbb{F}^2$  is given by  $b_\delta(v_i, v_i) = 2$  and  $b_\delta(v_0, v_1) = \delta \neq 2$  for its basis  $v_0, v_1$ .*
- (6)  *$\eta = \frac{1}{2}$  and  $A$  is isomorphic to the 2-dimensional special Jordan algebra  $\text{Cl}^0(\mathbb{F}^2, b_2)$  or the 3-dimensional Jordan algebra  $\text{Cl}^{00}(\mathbb{F}^2, b_2)$ , where the degenerate symmetric bilinear form  $b_2$  on  $\mathbb{F}^2$  is given by  $b_2(v_i, v_j) = 2$  for its basis  $v_0, v_1$ .*

For the definitions and discussion of the various examples, see Section 3.

The restriction (d) provides a  $\mathbb{Z}/2\mathbb{Z}$ -grading of  $A$  for each axis  $a$ . Equivalently, the linear transformation of  $A$  that acts as the identity on  $A_+(a)$  and negates everything in  $A_-(a)$  is an automorphism of  $A$ . The resulting automorphisms of order 2 will be called *Miyamoto involutions*, since in the Griess algebra context they were first noticed and used to great effect by Miyamoto in [Mi96]. Sakuma's proof and our proof of Theorem (1.1) make critical use of the dihedral group generated by the Miyamoto involutions corresponding to the two generators.

If  $a$  is an axis and  $g$  is an automorphism of  $A$ , then  $a^g$  is also an axis. For a generating set  $\mathcal{A}$  of axes, let  $\bar{\mathcal{A}}$  be the smallest set of axes with the properties:

- (i)  $\mathcal{A} \subseteq \bar{\mathcal{A}}$ .
- (ii) If  $b \in \bar{\mathcal{A}}$  and  $\tau$  is the Miyamoto involution associated with  $b$ , then  $\bar{\mathcal{A}}^\tau \subseteq \bar{\mathcal{A}}$ .

As a consequence of the theorem, every product of two members of  $\mathcal{A}$ , and indeed of  $\bar{\mathcal{A}}$ , is in the  $\mathbb{F}$ -span of  $\bar{\mathcal{A}}$ . Therefore

**(1.2).** COROLLARY. *Let  $A$  be an axial algebra of Jordan type  $\eta$  over a field  $\mathbb{F}$  of characteristic not two that is generated by the set  $\mathcal{A}$  of axes. Then  $A$  is spanned as  $\mathbb{F}$ -space by the axes of  $\bar{\mathcal{A}}$ .*

The theorem readily leads to

**(1.3).** THEOREM. *Let  $A$  be an axial algebra of Jordan type  $\eta \neq \frac{1}{2}$  over a field of characteristic not two that is generated by the set  $\mathcal{A}$  of axes. Then the Miyamoto involutions corresponding to  $\bar{\mathcal{A}}$  form a normal set of 3-transpositions in the automorphism group of  $A$  that they generate.*

For the definition and discussion of 3-transpositions, see Section 5.

The theorem and results from [CuHa95] imply that in finitely generated algebras with  $\eta \neq \frac{1}{2}$ , the set  $\bar{\mathcal{A}}$  is finite. Together with Corollary (1.2) this leads to

**(1.4).** COROLLARY. *If the axial algebra  $A$  of Jordan type  $\eta \neq \frac{1}{2}$  over the field  $\mathbb{F}$  of characteristic not two is finitely generated then it is finite dimensional as a vector space over  $\mathbb{F}$ .*

The theorem is in general false for Jordan type  $\eta = \frac{1}{2}$ , and we suspect that the corollary is also false in that case. Certainly in that case a finitely generated algebra can have  $\bar{\mathcal{A}}$  infinite; see remarks near the beginning of Section 5.

We have a converse to Theorem (1.3).

**(1.5).** THEOREM. *Let  $D$  be a normal set of 3-transpositions in the group  $G = \langle D \rangle$ . For any field  $\mathbb{F}$  not of characteristic two and any  $\eta \in \mathbb{F}$  with  $0 \neq \eta \neq 1$ , the space  $M = \mathbb{F}D$  can be given the structure of a primitive axial algebra of Jordan type  $\eta$  on which the elements of  $D$  act as Miyamoto involutions. The algebra  $M$  admits a nonzero symmetric and associative bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle: A \times A \rightarrow \mathbb{F}$ .*

This is proven in a more precise form in Theorem (6.4) and Corollary (7.4) below.

The historical context for the commutative algebras discussed in this paper has three separate branches. The first, discussed above, views axial algebras of Jordan type  $\eta$  as a first step away from semisimple, associative algebras—a step far enough away to include new and interesting examples,

such as the Jordan algebras generated by idempotents, but not so far as to defy meaningful classification.

The second branch, also mentioned above, was the actual motivation for the present work. In the early 1970's Bernd Fischer and Robert Griess independently found evidence for the existence of the Monster sporadic simple group  $\mathbb{M}$ . Soon after, it was noted that the smallest faithful  $\mathbb{R}$ -module for  $\mathbb{M}$  might well have dimension 196883, and Simon Norton observed that such a module would admit a commutative, nonassociative algebra structure [Gr76, CoNo79].

Bob Griess [Gr81] constructed this algebra and hence  $\mathbb{M}$  as an automorphism group (by hand). In the full treatment [Gr82] he preferred an algebra of dimension 196884, including a trivial  $\mathbb{M}$ -submodule. Conway [Co85] used a deformation  $B^\natural$  of the 196884 algebra to give a new construction. He noted the existence in  $B^\natural$  of idempotents (after appropriate scaling), one for each  $2A$  involution of the Monster, with adjoint minimal polynomial  $(x-1)x(x-\frac{1}{4})(x-\frac{1}{32})$ ; he called these idempotents *axial vectors*.

Motivated in part by the “Monstrous Moonshine” conjectures [CoNo79], Borchers [Bo86] codified vertex operators, and Frenkel, Lepowsky, and Meurman [FLM88] constructed a vertex operator algebra  $V^\natural$  whose graded piece  $V_2^\natural$  inherits from the VOA a natural commutative algebra structure isomorphic to  $B^\natural$ . The algebra  $V^\natural$  belongs to a large class of VOA for which  $V_2$  always admits a natural structure as commutative algebra. These commutative algebras are the *Griess algebras*, and Miyamoto [Mi96] observed that in them each conformal vector of central charge  $\frac{1}{2}$  can be viewed as an axis, in the sense that there is an involutory automorphism acting in a prescribed way relative to the  $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$ -eigenspaces of each of these conformal vectors. This effectively reverses Conway's construction of axes from the  $2A$  involutions of  $V_2^\natural = B^\natural$ . Miyamoto, Kitazume, and others [Mi96, KiMi01] then studied the possible groups that can be generated by these *Miyamoto involutions* within the automorphism groups of Griess algebras and their associated VOAs. Of particular importance for this paper is the work of Sakuma [Sa07], which described all groups generated by two such involutions, and that of Matsuo [Ma05], which completed the study of the case where only the eigenvalues  $\{1, 0, \frac{1}{4}\}$  occur. Indeed the original version [Ma03] of [Ma05] discussed the more general case of Griess algebras with axes admitting only three eigenvalues and noted there that the Miyamoto involutions are 3-transpositions, an observation due to Miyamoto [Mi96, Theorem 6.13] in the  $\{1, 0, \frac{1}{4}\}$  case. Matsuo's unpublished original thus contains versions of several of the main results of this paper, albeit in

the more restricted context of Griess algebras.

In an effort to divorce the properties of Griess algebras from the VOAs that envelope them, Ivanov [Iv09] introduced *Majorana algebras*. From our point of view (see Section 2) these, and so especially the Griess algebra examples, are real, Frobenius, primitive axial algebras with fusion table

$\star$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1,0	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	1,0, $\frac{1}{4}$

Ivanov, Pasechnik, Seress, and Shpectorov [IPSS10] extended Sakuma’s 2-generator theorem to Majorana algebras.

The third and final contextual branch for this paper begins with Simon Norton [No75, No88], who constructed a commutative, nonassociative algebra with automorphism group a triple cover of the sporadic group  $Fi_{24}$ . Although the term “Norton algebra” remains loosely defined [Sm77], such constructions start with a partial linear space of order 2 (see Section 6) as the skeleton for a presentation of a commutative algebra via its 2-generated subalgebras. Work along these lines was done by many, in part to characterize various groups as the automorphism groups of algebra structures on related “natural” modules. A case in point is the paper [Har84] in which, among other things, Harada constructed a 6-dimensional commutative nonassociative algebra with automorphism group  $SL_3(2)$ , using as skeleton  $PG(2, 2)$ —the projective plane of order 2. The unpublished [MaMa99] of Matsuo and Matsuo studied in detail a related 1-parameter family of complex 7-dimensional algebras, each with skeleton  $PG(2, 2)$ , generically having  $SL_3(2)$  as automorphism group. The unpublished original [Ma03] pursued and generalized this construction, noting that for Griess algebras the appropriate partial linear spaces to consider are the Fischer spaces—the geometric counterparts to 3-transposition groups. The algebras of [MaMa99] and [Ma03] are examples of the *Matsuo algebras* introduced here in Section 6. Those corresponding to Fischer spaces were also discussed in [Re13].

One possible path for generalization is to replace the idempotence condition by the requirement that  $a^2 = ka$ , for some constant  $k$ . If  $k$  is a nonzero field element then rescaling gets us back to the idempotent case, but  $a^2 = 0$

can lead to interesting algebras. Let  $M(\Pi, \frac{1}{2}, \mathbb{Q})$  be the rational Matsuo algebra for the partial linear space  $\Pi$  of order 2, and let  $A$  be its  $\mathbb{Z}$ -subalgebra spanned the various elements  $2a_p$ , for  $p$  a point of  $\Pi$ . Then  $A/2A$  is a commutative, nonassociative  $\mathbb{F}_2$ -algebra generated by elements with square 0. For various choices of  $\Pi$  these algebras and their quotients give rise to Lie algebras and “near-Lie” algebras that have interesting automorphism groups, as can be seen in [Cu05] and [CHpS12].

We conclude with a brief outline of the paper. After this introduction we present the basic definitions for axial algebras and their related fusion rules. The fundamental objects are the semisimple idempotents, and we include a characterization of semisimple associative commutative algebras in this context. Of particular interest are the algebras for which the fusion rules guarantee the existence of Miyamoto involutory automorphisms.

In the algebras of Jordan type the idempotents have at most three adjoint eigenvalues—1 and 0 (as expected) and some  $\eta \neq 0, 1$ . Section 3 describes various examples of primitive axial algebras of Jordan type. The focus is on 2-generated algebras, but the special case of Jordan algebras is discussed in greater generality.

Section 4 contains the proof of Theorem (1.1) of Sakuma type. The next section then describes the impact of the Sakuma theorem on the automorphism group of an arbitrary primitive axial algebra of Jordan type  $\eta \neq \frac{1}{2}$ . Specifically the Miyamoto involutions provide a normal set of 3-transpositions, as detailed in Theorem (1.3). The classifications of [Fi71] and [CuHa95] are then available.

In Section 6 we provide a constructional converse to the results of the previous section. Beginning with a partial linear space of order 2, an algebra is constructed—the *Matsuo algebra*. In particular it is proven, as in Theorem (1.5), that every normal set of 3-transpositions arises as a set of Miyamoto involutions for an appropriate primitive axial algebra of Jordan type  $\eta$ , for arbitrary  $\eta$ . Indeed the Matsuo algebras that have Jordan type  $\eta$  are precisely those with Fischer spaces as skeletons; see Theorem (6.5).

Griess algebras come equipped with an associative form, and in the final Section 7 we describe the circumstances under which a Matsuo algebra admits such a form. This includes all the axial algebras of Theorem (1.5).

Parts of the present article have much in common with the unpublished paper of Matsuo [Ma03] (although he makes some assumptions not made here—he assumes characteristic 0 and proves a version of Theorem (1.1) only for Griess algebras). We warmly thank Professor Matsuo for pointing out [Ma03] and providing us with hard copy of the otherwise unavail-

able [MaMa99].

## 2 Fusion and axial algebras

### 2.1 Decomposition, fusion, and grading

For the  $\mathbb{F}$ -algebra  $A$  and set  $I$ , a *decomposition* results from writing  $A$  as a direct sum of subspaces indexed by  $I$ :

$$A = \bigoplus_{i \in I} A_i.$$

Each of our decompositions will be accompanied by a *fusion rule*

$$\mathfrak{F}: I \times I \longrightarrow 2^I,$$

a map that takes each ordered pair  $i, j$  of indices to a member  $i \star j$  of the power set of  $I$  and that encodes the *fusion* information

$$A_i A_j \subseteq \bigoplus_{k \in i \star j} A_k.$$

We then have an  $\mathfrak{F}$ -*decomposition*.

Of course every decomposition admits the trivial fusion rule with  $i \star j = I$ , for all  $i, j \in I$ . Equally well every decomposition has a unique *minimal* fusion rule, where each  $i \star j$  is chosen with cardinality as small as possible.

We are interested in rules where all  $i \star j$  have small cardinality, in which case the fusion rule may be easily presented in a *fusion table*. In particular, if each  $i \star j$  has cardinality 1, then the fusion table gives the multiplication table for a magma  $(I, \star)$ , and the encoded fusion properties describe a *grading* of  $A$  by that magma.

The algebras we study will be graded by  $\mathbb{Z}/2\mathbb{Z}$ , the cyclic group of order two. We there take  $I = \{\pm 1\} = \{\pm\}$  and  $i \star j = ij$ . It is well-known and easy to see that in characteristic not two the existence of a  $\mathbb{Z}/2\mathbb{Z}$ -grading is equivalent to the existence of an automorphism of order (at most) two:

**(2.1). PROPOSITION.** *Let  $A$  be an  $\mathbb{F}$ -algebra with the characteristic of  $\mathbb{F}$  not two. Suppose  $A = A_+ \oplus A_-$ . Then the following are equivalent:*

- (1) *The indices give a  $\mathbb{Z}/2\mathbb{Z}$ -grading of  $A$ .*
- (2)  *$A$  has an automorphism of order at most two that fixes each element of  $A_+$  and takes each element of  $A_-$  to its negative.*

*The automorphism has order 1 precisely when  $A_+ = A$  and  $A_- = 0$ . □*



## 2.2 Semisimple idempotents

As mentioned above, for  $a \in A$  and  $\lambda \in \mathbb{F}$ , the  $\lambda$ -eigenspace for the adjoint  $\text{ad}_a$  will be denoted  $A_\lambda(a)$  (allowing  $A_\lambda(a) = 0$ ).

If  $A$  is associative and  $a$  an idempotent, then  $A = A_1(a) \oplus A_0(a)$ . If  $A$  is not associative, then its idempotents can have adjoint eigenvalues other than 1 and 0 and the minimal polynomial of the adjoint need not be squarefree. A *Peirce decomposition* for  $A$  with respect to the *semisimple idempotent* or *axis*  $a$  is a decomposition

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda(a),$$

where  $\Lambda \subseteq \mathbb{F}$  is a set containing all eigenvalues for the adjoint action of  $a$  on the algebra  $A$ .

In particular, Griess and Majorana algebras provide commutative algebras (typically over  $\mathbb{R}$ ) generated by axes whose corresponding Peirce decompositions have fusion table:

$\star$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1,0	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	1,0, $\frac{1}{4}$

It is usual [Iv09, p.210] to take  $1 \star 0 = 0$  in these rules, but primitivity of the algebra (see (2.2) below) allows the stronger  $1 \star 0 = \emptyset$ , which we prefer.

The axes  $a$  in these algebras have certain properties that will be of general interest:

**(2.2).** HYPOTHESIS. (Primitivity)  $A_1(a) = \mathbb{F}a$ .

**(2.3).** HYPOTHESIS. (Seress Condition) For  $\lambda \neq 1$ ,  $A_\lambda(a)A_0(a) \subseteq A_\lambda(a)$

The Seress Condition implies the weaker

**(2.4).** HYPOTHESIS. (0-subalgebra)  $A_0(a)$  is a subalgebra.

**(2.5).** HYPOTHESIS. ( $\mathbb{Z}/2\mathbb{Z}$ -grading) The eigenvalue set  $\Lambda$  is the disjoint union of  $\Lambda_+$  and  $\Lambda_-$  with  $1 \in \Lambda_+$  and such that

$$A_+ = \bigoplus_{\lambda \in \Lambda_+} A_\lambda(a) \quad \text{and} \quad A_- = \bigoplus_{\lambda \in \Lambda_-} A_\lambda(a)$$

provides a  $\mathbb{Z}/2\mathbb{Z}$ -grading of  $A$ .

For example, the axes  $a$  above have a fusion rule that is a refinement of the  $\mathbb{Z}/2\mathbb{Z}$ -grading

$$A_+(a) = A_1(a) \oplus A_0(a) \oplus A_{\frac{1}{4}}(a) \quad \text{and} \quad A_-(a) = A_{\frac{1}{32}}(a).$$

Furthermore, when  $A_{\frac{1}{32}}(a) = 0$  we have the alternative  $\mathbb{Z}/2\mathbb{Z}$ -grading

$$A_+(a) = A_1(a) \oplus A_0(a) \quad \text{and} \quad A_-(a) = A_{\frac{1}{4}}(a).$$

For the axis  $a$  having a  $\mathbb{Z}/2\mathbb{Z}$ -grading as in (2.5), we define  $\tau(a)$  to be the linear transformation of  $A$  that acts as the identity on  $A_+(a)$  and negates everything in  $A_-(a)$ . By Proposition (2.1) these are automorphisms. Those of order 2 are the *Miyamoto involutions* of  $A$ , since in the Griess algebra context they were introduced by Miyamoto [Mi96]. There they were called  $\sigma$ - or  $\tau$ -involutions depending, respectively, upon whether  $A_{\frac{1}{32}}(a)$  was zero or not. In the Majorana algebra case these are the *Majorana involutions* of Ivanov [Iv09]. The  $A_{\frac{1}{32}}(a) = 0$  case is related to the *Atkin-Lehner involutions* of Norton [No96, Theorem 3].

### 2.3 Axial algebras

An *axial algebra* is a commutative algebra that is generated by a set of axes. This has little force unless the axes are provided with rigid fusion properties, as is the case with the Griess and Majorana algebras, whose fusion table appears above.

The Griess and Majorana algebra examples are *primitive axial algebras*, in that for each generating axis  $a$  we have  $A_1(a) = \mathbb{F}a$ , as in (2.2).

Generally, the fusion rule (and table) for the axial algebra  $A$  generated by the axis set  $\mathcal{A}$  has index set  $\Lambda$ , the union of the eigenvalue sets for the individual  $a \in \mathcal{A}$ , and each  $\mu \star \nu$  satisfies

$$A_\mu(a)A_\nu(a) \subseteq \bigoplus_{\lambda \in \mu \star \nu} A_\lambda(a),$$

for all  $a \in \mathcal{A}$ .

The fusion rule given above for the Griess and Majorana algebras is actually the particular example  $\mathfrak{V}(4, 3)$  from the class of *Virasoro fusion rules*  $\mathfrak{V}(p, q)$ , which arise from the representation theory of rational Virasoro algebras. (See [HRS13] for a more detailed discussion.)

Virasoro rules are associated with vertex operator algebras, and the corresponding Griess algebras come equipped with a form satisfying the Frobenius property:

**(2.6).** HYPOTHESIS. (Frobenius Property) *There is a bilinear form*

$$\langle\langle \cdot, \cdot \rangle\rangle : A \times A \longrightarrow \mathbb{F}$$

*with  $\langle\langle a, a \rangle\rangle \neq 0$ , for all  $a \in \mathcal{A}$ , that is associative:*

$$\langle\langle ax, b \rangle\rangle = \langle\langle a, xb \rangle\rangle$$

*for all  $a, b, x \in A$ .*

For Griess algebras this form is in fact positive definite, but that supposes characteristic zero and will not appear again until the very end of the paper.

Frobenius axial algebras have some useful properties.

**(2.7).** PROPOSITION. *Let  $A$  be a Frobenius axial algebra.*

- (a) *The form  $\langle\langle \cdot, \cdot \rangle\rangle$  is symmetric.*
- (b) *For axis  $a$  and distinct  $\lambda, \mu$ , the eigenspaces  $A_\lambda(a)$  and  $A_\mu(a)$  are perpendicular.*
- (c) *If  $A$  is primitive, then the radical of the form is the unique largest ideal of  $A$  that does not contain any of the generating axes.*

PROOF. The first two are elementary and can be found in [HRS13, Prop. 3.5-6].

Each ideal is the direct sum of its eigenspaces for any given axis. Therefore by (b), if an ideal of a primitive axial algebra does not contain any of the generating axes, it is perpendicular to those axes. In view of associativity of the form, it now follows that the ideal is perpendicular to any product of the generating axes and so is in the radical  $R$ .

On the other hand, for all  $a, x \in A$  and  $r \in R$

$$0 = \langle\langle ax, r \rangle\rangle = \langle\langle a, xr \rangle\rangle.$$

Thus the radical  $R$  is itself an ideal and so the maximal ideal containing none of the axes.  $\square$

The associative algebra  $\bigoplus_{i \in I} \mathbb{F}a_i$  is a primitive Frobenius axial algebra generated by the primitive orthogonal idempotents  $\{a_i \mid i \in I\}$  and having fusion table

$\star$	$1$	$0$
$1$	$1$	$\emptyset$
$0$	$\emptyset$	$0$

as  $a_i a_j = 0$  for  $i \neq j$ .

We can characterize these associative algebras in axial terms.

**(2.8). PROPOSITION.** *Let  $A$  be a primitive axial algebra over the field  $\mathbb{F}$  that is generated by the set  $\mathcal{A}$  of axes. Then for each  $a \in \mathcal{A}$*

$$(\mathbb{F}a \oplus A_0(a)) \cap \mathcal{A} = \{a\} \cup (A_0(a) \cap \mathcal{A}).$$

PROOF. Let  $b \in (\mathbb{F}a \oplus A_0(a)) \cap \mathcal{A}$ , and set  $b = \mu a + n$  for  $\mu \in \mathbb{F}$  and  $n \in A_0(a)$ . Then

$$\mu a + n = b = b^2 = (\mu a + n)^2 = \mu^2 a + n^2,$$

hence  $n = n^2$  and  $\mu = \mu^2$ . That is, either  $\mu = 0$  and  $n = b \in A_0(a)$ , as desired, or  $\mu = 1$ .

Suppose that  $\mu = 1$  and  $b = a + n$ . Therefore

$$ab = a(a + n) = a^2 + an = a \in A_1(b).$$

But by the primitivity assumption  $A_1(b) = \mathbb{F}b$  contains the unique idempotent  $b = a$ .  $\square$

**(2.9). COROLLARY.** *Let  $A$  be a primitive axial algebra over the field  $\mathbb{F}$  that is generated by the set  $\mathcal{A}$  of axes having fusion table*

$\star$	$1$	$0$
$1$	$1$	$\emptyset$
$0$	$\emptyset$	$0$

*Then  $A = \bigoplus_{a \in \mathcal{A}} \mathbb{F}a$  is associative.*

PROOF. For each  $a \in \mathcal{A}$ , the algebra  $A$  is the direct sum of  $\mathbb{F}a$  and the subalgebra of  $A_0(a)$  generated by all the axes of  $\mathcal{A}$  except for  $a$ .  $\square$

In this article we are mainly interested in a case where the algebras are minimally nonassociative. For  $\eta \in \mathbb{F}$  with  $0 \neq \eta \neq 1$ , the axial  $\mathbb{F}$ -algebra  $A$  is said to have *Jordan type  $\eta$*  when it is generated by a set of axes with fusion table:

$\star$	1	0	$\eta$
1	1	$\emptyset$	$\eta$
0	$\emptyset$	0	$\eta$
$\eta$	$\eta$	$\eta$	1, 0

In particular, the algebras of Jordan type have a  $\mathbb{Z}/2\mathbb{Z}$ -grading and the Seress Condition. Indeed for primitive algebras the weaker 0-subalgebra condition suffices.

**(2.10). LEMMA.** *The primitive axial algebra  $A$  has Jordan type  $\eta$  ( $\neq 0, 1$ ) if and only if it has the following two properties for each  $a$  in its generating axis set:*

- (i) (0-subalgebra)  $A_0(a)$  is a subalgebra.
- (ii) ( $\mathbb{Z}/2\mathbb{Z}$ -grading)  $A_+(a) = A_1(a) \oplus A_0(a)$  and  $A_-(a) = A_\eta(a)$  provide a  $\mathbb{Z}/2\mathbb{Z}$ -grading of  $A$ .  $\square$

Thus the primitive axial algebras of Jordan type  $\eta$  are exactly the commutative algebras generated by a set of idempotents having the properties (a)-(d) of the introduction.

### 3 Examples

**(3.1).** An axial algebra over  $\mathbb{F}$  generated by a single axis is isomorphic to  $\mathbb{F}$  with the axis  $z_0 = 1$ . This algebra is denoted 1A and only contains one idempotent, namely its identity element.

**(3.2).** The 2-dimensional associative algebra  $\mathbb{F} \oplus \mathbb{F}$  with  $\{b_0, b_1\}$  as  $\mathbb{F}$ -basis and having relations

$$b_0^2 = b_0, \quad b_1^2 = b_1, \quad b_0 b_1 = 0$$

is an axial algebra with axes  $\{b_0, b_1\}$  and has Jordan type  $\eta$  for all  $\eta$ . This algebra is denoted 2B. Its only idempotents, other than  $b_0$  and  $b_1$ , are the identity element  $b_0 + b_1$  and 0.

**(3.3).** For  $\eta \in \mathbb{F}$  let the  $\mathbb{F}$ -algebra  $3C(\eta)$  have basis  $\{c_0, c_1, c_2\}$  and be subject (for  $\{i, j, k\} = \{0, 1, 2\}$ ) to relations

$$c_i^2 = c_i, \quad c_i c_j = \frac{\eta}{2}(c_i + c_j - c_k).$$

Then

$$\begin{aligned}
c_i(\eta c_i - c_j - c_k) &= \eta c_i^2 - \frac{\eta}{2}(c_i + c_j - c_k) - \frac{\eta}{2}(c_i + c_k - c_j) \\
&= \eta c_i - \frac{\eta}{2}c_i - \frac{\eta}{2}c_j + \frac{\eta}{2}c_k - \frac{\eta}{2}c_i - \frac{\eta}{2}c_k + \frac{\eta}{2}c_j \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
c_i(c_j - c_k) &= \frac{\eta}{2}(c_i + c_j - c_k) - \frac{\eta}{2}(c_i + c_k - c_j) \\
&= \frac{\eta}{2}c_i + \frac{\eta}{2}c_j - \frac{\eta}{2}c_k - \frac{\eta}{2}c_i - \frac{\eta}{2}c_k + \frac{\eta}{2}c_j \\
&= \eta(c_j - c_k).
\end{aligned}$$

Thus for  $\eta \neq 0, 1$ , the algebra of type  $3C(\eta)$  over  $\mathbb{F}$  is a 3-dimensional axial algebra with axis set  $\{c_0, c_1, c_2\}$  as an  $\mathbb{F}$ -basis and generated by  $c_0, c_1$ .

Comparing this definition with [IPSS10, Table 4], we see that the algebra  $3C(\frac{1}{32})$  over  $\mathbb{R}$  is the 3C Majorana algebra, and  $3C(\frac{1}{4})$  over  $\mathbb{R}$  is the 2A Majorana algebra.

The axial algebra  $3C(\eta)$  is, in fact, of Jordan type  $\eta$ . For each of the axes  $c_i$ , the map that fixes  $c_i$  and switches  $c_j$  and  $c_k$  clearly induces an automorphism. The above calculations show that this is the Miyamoto involution  $\tau(c_i)$ , so we have the needed  $\mathbb{Z}/2\mathbb{Z}$ -grading. It remains to prove that the 0-eigenspace for  $\text{ad}_{c_i}$  is a subalgebra:

$$\begin{aligned}
(\eta c_i - c_j - c_k)(\eta c_i - c_j - c_k) &= (-c_j - c_k)(\eta c_i - c_j - c_k) \\
&= -\eta c_j c_i - \eta c_k c_i + 2c_j c_k + c_j + c_k \\
&= -\frac{\eta^2}{2}(c_j + c_i - c_k) - \frac{\eta^2}{2}(c_k + c_i - c_j) \\
&\quad + \eta(c_j + c_k - c_i) + c_j + c_k \\
&= -(\eta^2 + \eta)c_i + (\eta + 1)c_j + (\eta + 1)c_k \\
&= -(\eta + 1)(\eta c_i - c_j - c_k),
\end{aligned}$$

as desired.

The algebra  $3C(\eta)$  can contain other idempotents, nevertheless  $\bar{\mathcal{A}} = \{c_0, c_1, c_2\}$ . It is also a Frobenius algebra, the symmetric associative bilinear form (unique up to scalar multiple) being given by

$$\langle\langle c_i, c_i \rangle\rangle = 1, \quad \langle\langle c_i, c_j \rangle\rangle = \frac{\eta}{2}.$$

See Lemma (7.1) below.

The values  $\eta = 0, 1$  are indeed exceptional. The algebra  $3C(0)$  is a copy of the associative algebra  $\mathbb{F}^3$  of type  $1A^3$  and is not two-generated. The algebra  $3C(1)$  has 1-eigenspaces of dimension 2 for its idempotents  $c_i$ . It is  $\mathbb{F}^3$  again, but the idempotents  $c_i$  are now the elements of  $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ .

**(3.4).** In the special case  $\eta = -1$ , the algebra  $3C(-1)$  has the radical ideal  $\mathbb{F}(c_0 + c_1 + c_2)$ . Let  $3C(-1)^*$  be the quotient by this ideal, and set  $d_i = c_i + \mathbb{F}(c_0 + c_1 + c_2)$ . This is a 2-dimensional axial algebra spanned by any two of its three axes  $\{d_0, d_1, d_2 (= -d_0 - d_1)\}$  and having multiplication given by (for  $\{i, j, k\} = \{0, 1, 2\}$ )

$$d_i^2 = d_i, \quad d_i d_j = d_k.$$

As the ideal  $\mathbb{F}(c_0 + c_1 + c_2)$  in  $3C(-1)$  is the radical,  $3C(-1)^*$  remains a Frobenius algebra.<sup>1</sup>

**(3.5).** A *Jordan algebra* [Ja68] over  $\mathbb{F}$  (of characteristic not two) is a commutative  $\mathbb{F}$ -algebra satisfying the identical relation

$$(a^2 b)a = a^2 (ba).$$

The basic example of a Jordan algebra begins with an associative algebra  $(A, \cdot, +)$  and defines on  $A$  a new multiplication (the *Jordan product*)

$$a \circ b = \frac{1}{2}(ab + ba);$$

the algebra  $(A, \circ, +)$  is then a Jordan algebra, typically denoted  $A^+$ . Those Jordan algebras that arise as subalgebras of  $A^+$  for some associative algebra  $A$  are called *special*.

We are specifically interested in the Clifford algebra  $\text{Cl}(V, b)$  of the symmetric bilinear form  $b$  defined on the  $\mathbb{F}$ -space  $V$ . (See [Ja68, p.75] for discussion of Clifford algebras.)  $\text{Cl}(V, b)$  is the associative  $\mathbb{F}$ -algebra that results from factoring out of the tensor algebra  $T(V)$  the ideal generated by all elements  $v \otimes w + w \otimes v - b(v, w)1$  for  $v, w \in V$ . As  $v \otimes v - \frac{1}{2}b(v, v)$  is in the ideal, the Clifford algebra is spanned by the various monomials  $\prod_{i=1}^k v_{\iota(i)}$ ,

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<sup>1</sup>Alberto Elduque observed that  $3C(-1)^*$  has a natural interpretation as para-Hurwitz algebra: depending upon whether or not  $\mathbb{F}$  contains a primitive cube root of unity  $\omega$ , let  $H$  be the Hurwitz algebra  $\mathbb{F} \oplus \mathbb{F}$  or  $\mathbb{F}(\omega)$ ; then  $3C(-1)^*$  is the para-Hurwitz algebra obtained by replacing the usual multiplication  $\cdot$  for  $H$  by  $a \star b = (a \cdot b)^J$  where  $J$  is conjugation in  $H$ . Jens Köpplinger noted that for  $\mathbb{F} \leq \mathbb{R}$  this para-Hurwitz algebra and its associated form can be described nicely in terms of the  $A_2$  root lattice thought of a subset of the complex plane. We thank both for their remarks.

for nonnegative  $k$ , where  $\iota$  is an injection of  $1, \dots, k$  into the index set  $I$  for the basis  $\{v_i \mid i \in I\}$  of  $V$ . The Clifford algebra admits the canonical reversal involution  $J$  determined by

$$J: \prod_{i=1}^k v_{\iota(i)} \mapsto \prod_{i=k}^1 v_{\iota(i)}$$

for all  $k$  and all  $k$ -subsets  $\{v_{\iota(i)} \mid 1 \leq i \leq k\}$ . Its fixed points form  $\text{Cl}^J(V, b)$ , a Jordan subalgebra of  $\text{Cl}(V, b)^+$ .

The special Jordan algebra  $\text{Cl}^J(V, b)$  of  $J$ -symmetric Clifford elements in  $\text{Cl}(V, b)$  contains the subset  $V^J(b) = \mathbb{F}1 \oplus V$  which is in fact a (special) Jordan subalgebra itself because

$$\begin{aligned} (\alpha 1 + v) \circ (\beta 1 + w) &= \frac{1}{2} ((\alpha 1 + v)(\beta 1 + w) + (\beta 1 + w)(\alpha 1 + v)) \\ &= \frac{1}{2} (\alpha\beta 1 + \alpha w + \beta v + vw + \beta\alpha 1 + \beta v + \alpha w + wv) \\ &= \frac{1}{2} (2\alpha\beta 1 + 2\alpha w + 2\beta v + vw + wv) \\ &= \frac{1}{2} ((2\alpha\beta + b(v, w))1 + 2\alpha w + 2\beta v) \\ &= (\alpha\beta + \frac{1}{2}b(v, w))1 + \alpha w + \beta v. \end{aligned}$$

We collect some properties of the Jordan algebra  $V^J$ :

- (a) *The nonidentity idempotents of  $V^J$  are exactly the elements  $\frac{1}{2} + v$  for  $v \in V$  with  $b(v, v) = \frac{1}{2}$ .*
- (b) *If  $e^+ = \frac{1}{2} + v$  is a nonidentity idempotent, then so is  $e^- = \frac{1}{2} - v$ . Furthermore for  $\epsilon = \pm$*

$$V_1^J(e^\epsilon) = \mathbb{F}e^\epsilon, \quad V_0^J(e^\epsilon) = \mathbb{F}e^{-\epsilon},$$

and

$$V_{\frac{1}{2}}^J(e^+) = V_{\frac{1}{2}}^J(e^-) = v^\perp.$$

*In particular the idempotent  $e^\epsilon$  is semisimple with eigenvalues  $\{1, 0, \frac{1}{2}\}$  and 1- and 0-eigenspaces of dimension 1.*

- (c) *For  $e^\epsilon = \frac{1}{2} + \epsilon v$  ( $\epsilon = \pm$ ) a nonidentity idempotent, let  $t$  with  $t(1) = 1$  be the extension to  $V^J$  of the negative of the orthogonal reflection on  $V$  with center  $v$ . Then  $t$  is the Miyamoto involution  $\tau(e^+) = \tau(e^-)$  on  $V^J$ . In particular,  $V^J$  is  $\mathbb{Z}/2\mathbb{Z}$ -graded by  $e^\epsilon$  with  $\Lambda_+ = \{1, 0\}$  and  $\Lambda_- = \{\frac{1}{2}\}$ .*



PROOF. We identify  $\mathbb{F}$  and  $\mathbb{F}1$ . Also we no longer use  $\circ$  to indicate multiplication in  $V^J$ , which is thus given by

$$(\alpha + v)(\beta + w) = (\alpha\beta + \frac{1}{2}b(v, w)) + \alpha w + \beta v.$$

(a) We have

$$(\alpha + v)^2 = (\alpha^2 + \frac{1}{2}b(v, v)) + 2\alpha v.$$

For idempotent  $\alpha + v$  this forces  $\alpha = \frac{1}{2}$  and then  $b(v, v) = \frac{1}{2}$ .

(b) As  $b(v, v) = b(-v, -v)$ ,  $e^+$  is an idempotent if and only if  $e^-$  is, by the previous part. Next

$$e^\epsilon(\beta + w) = (\frac{1}{2} + \epsilon v)(\beta + w) = (\frac{\beta}{2} + \frac{\epsilon}{2}b(v, w)) + \frac{1}{2}w + \beta\epsilon v.$$

For  $\beta + w$  to be in  $V_1^J(e^+)$  we must have  $\frac{1}{2}w + \beta\epsilon v = w$ ; that is,  $w = 2\beta\epsilon v$ . Therefore

$$V_1^J(e^\epsilon) = \{\beta + 2\beta\epsilon v\} = \{2\beta(\frac{1}{2} + \epsilon v)\} = \mathbb{F}e^\epsilon.$$

Similarly if  $\beta + w \in V_0^J(e^\epsilon)$  then  $\frac{1}{2}w + \beta\epsilon v = 0$  and  $w = -2\beta\epsilon v$ , leading to  $V_0^J(e^\epsilon) = \mathbb{F}e^{-\epsilon}$ .

Let  $W = v^\perp$ , the subspace of  $V$  consisting of elements that are  $b$ -perpendicular to  $v$ . Then

$$V^J = \mathbb{F}e^+ \oplus \mathbb{F}e^- \oplus W = \mathbb{F}1 \oplus \mathbb{F}v \oplus W.$$

Then for  $w \in W$

$$e^\epsilon w = (\frac{0}{2} + \frac{\epsilon}{2}b(v, w)) + \frac{1}{2}w + 0 \cdot \epsilon v = \frac{1}{2}w,$$

so that  $W = V_{\frac{1}{2}}^J(e^\epsilon)$  and

$$V^J = V_1^J(e^\epsilon) \oplus V_0^J(e^\epsilon) \oplus V_{\frac{1}{2}}^J(e^\epsilon).$$

(c) The negative reflection in  $v$  is an isometry of the form  $b$  on  $V$ ; so its extension  $t$  to  $V^J$  is an algebra automorphism, which by the previous part is equal to both Miyamoto involutions  $\tau(e^+)$  and  $\tau(e^-)$ . The existence of this automorphism is equivalent to  $\mathbb{Z}/2\mathbb{Z}$ -grading.  $\square$

Consider the symmetric Jordan Clifford algebra  $\text{Cl}^J(\mathbb{F}^2, b_\delta)$  on the vector space  $\mathbb{F}^2$  with basis  $v_0, v_1$  and symmetric bilinear form given by  $b_\delta(v_i, v_i) = 2$ ,  $b_\delta(v_0, v_1) = \delta$  for some constant  $\delta \in \mathbb{F}$ . (Note that different choices of  $\delta$  may give isometric spaces and so isomorphic algebras.)

As  $V$  has dimension 2, its Clifford algebra is spanned by  $1, v_0, v_1, v_0v_1$ ; so the Jordan algebra  $\text{Cl}^J(\mathbb{F}^2, b_\delta)$  is equal to its subalgebra  $V^J$  with basis  $1, v_0, v_1$ .

This is an axial algebra with generating axes

$$e_0^+ = \frac{1}{2}(1 + v_0), \quad e_0^- = \frac{1}{2}(1 - v_0), \quad e_1^+ = \frac{1}{2}(1 + v_1), \quad e_1^- = \frac{1}{2}(1 - v_1),$$

or, in fact, any three of these. Furthermore

$$\begin{aligned} e_0^+ e_1^+ &= \frac{1}{2}(1 + v_0) \frac{1}{2}(1 + v_1) \\ &= \frac{1}{4}(1 + v_0 + v_1 + v_0v_1) \\ &= \frac{1}{4}(1 + v_0 + v_1 + \frac{1}{2}b(v_0, v_1)) \\ &= \frac{1}{4}(1 + v_0 + v_1 + \frac{\delta}{2}) \\ &= \frac{1}{4}(1 + v_0 + 1 + v_1 + (\frac{\delta}{2} - 1)) \\ &= \frac{1}{2}e_0^+ + \frac{1}{2}e_1^+ + \frac{1}{8}(\delta - 2)1; \end{aligned}$$

so  $V^J$  is generated by  $e_0^+$  and  $e_1^+$  except when  $\delta = 2$ .

Every Jordan algebra generated by idempotents is a Frobenius algebra [Ja68]. Here that is easy to check for the form given by

$\langle\langle \cdot, \cdot \rangle\rangle$	1	$v_0$	$v_1$
1	2	0	0
$v_0$	0	2	$\delta$
$v_1$	0	$\delta$	2

That is, the form  $b_\delta$  on  $V$  is extended to the full algebra  $\mathbb{F}1 \oplus V$  by setting  $\langle\langle 1, 1 \rangle\rangle = 2$  and  $V = 1^\perp$ .

For  $\delta = 2$ , the idempotents  $e_0^+$  and  $e_1^+$  generate a 2-dimensional Jordan subalgebra  $\text{Cl}^0(\mathbb{F}^2, b_2)$  that is of type 1A when factored by the 1-dimensional ideal spanned by  $e_1^+ - e_0^+ = \frac{1}{2}(v_1 - v_0)$ .

In any event we have the multiplication table:

$\cdot$	$e_0^+$	$e_1^+$	$s$
$e_0^+$	$e_0^+$	$\frac{1}{2}e_0^+ + \frac{1}{2}e_1^+ + s$	$pe_0^+$
$e_1^+$	$\frac{1}{2}e_0^+ + \frac{1}{2}e_1^+ + s$	$e_1^+$	$pe_1^+$
$s$	$pe_0^+$	$pe_1^+$	$ps$

for  $p = \frac{1}{8}(\delta - 2) \in \mathbb{F}$  and  $s = \frac{1}{8}(\delta - 2)1 \in V^J$ .

Isometric forms  $b_\delta, b_{\delta'}$  produce isomorphic Clifford algebras, hence isomorphic symmetric Jordan Clifford algebras  $\text{Cl}^J(\mathbb{F}^2, b_*)$ . For instance, the algebras  $\text{Cl}^J(\mathbb{F}^2, b_\delta)$  and  $\text{Cl}^J(\mathbb{F}^2, b_{-\delta})$  are isomorphic as they correspond, respectively, to the bases  $v_0, v_1$  and  $v'_0 = v_0, v'_1 = -v_1$ . This amounts to replacing the pair of idempotents  $e_0^+, e_1^+$  by the pair  $(e'_0)^+ = e_0^+, (e'_1)^+ = e_1^-$ . The above multiplication table makes it clear that no automorphism of  $\text{Cl}^J(\mathbb{F}^2, b_\delta)$  takes the pair of axes  $\mathcal{A} = \{e_0^+, e_1^+\}$  to the pair  $\mathcal{A}' = \{(e'_0)^+ = e_0^+, (e'_1)^+ = e_1^-\}$ .

Indeed in the exceptional degenerate case  $\delta = 2$ , the axis set  $\mathcal{A}$  does not even generate  $\text{Cl}^J(\mathbb{F}^2, b_\delta)$  (as noted above) while  $\mathcal{A}'$  does.

**(3.6).** In the previous example, when  $\delta = 2$  the element  $s$  equals 0 since the scalar  $p = \frac{1}{2}(\delta - 2)$  is 0. But we can adapt the corresponding multiplication table to include that case. The algebra  $\text{Cl}^{00}(\mathbb{F}^2, b_2)$  is the vector space  $\mathbb{F}e'_0 \oplus \mathbb{F}e'_1 \oplus \mathbb{F}s'$  subject to the multiplication table:

$\cdot$	$e'_0$	$e'_1$	$s'$
$e'_0$	$e'_0$	$\frac{1}{2}e'_0 + \frac{1}{2}e'_1 + s'$	0
$e'_1$	$\frac{1}{2}e'_0 + \frac{1}{2}e'_1 + s'$	$e'_1$	0
$s'$	0	0	0

This is a Jordan algebra whose ideal  $\mathbb{F}s'$  is the 0-eigenspace for both of the generating idempotents  $e'_0$  and  $e'_1$ . The quotient by this ideal is a copy of  $\text{Cl}^0(\mathbb{F}^2, b_2)$ .

## 4 Two-generated axial algebras of Jordan type

Let  $A$  be a primitive axial algebra of Jordan type  $\eta$  over  $\mathbb{F}$  that is generated by the two axes  $a$  and  $b$ .

For  $x \in A$  and  $p \in \mathcal{A} = \{a, b\}$ , write

$$x = \varphi_p(x)p + \alpha_p(x) + \gamma_p(x)$$

for  $\varphi_p(x) \in \mathbb{F}$ ,  $\alpha_p(x) \in A_0(p)$ , and  $\gamma_p(x) \in A_\eta(p)$ .

Recall that the Miyamoto involutions  $\tau(p)$  are automorphisms that act via

$$x^{\tau(p)} = \varphi_p(x)p + \alpha_p(x) - \gamma_p(x).$$

**(4.1). LEMMA.**

- (a)  $\gamma_b(a) = \frac{1}{2}(a - a^{\tau(b)})$  and  $\gamma_a(b) = \frac{1}{2}(b - b^{\tau(a)})$ .
- (b)  $\alpha_b(a) = -\varphi_b(a)b + \frac{1}{2}(a + a^{\tau(b)})$  and  $\alpha_a(b) = -\varphi_a(b)a + \frac{1}{2}(b + b^{\tau(a)})$ .
- (c)  $ab = \varphi_b(a)b + \eta\gamma_b(a)$  and  $ab = \varphi_a(b)a + \eta\gamma_a(b)$ .
- (d) Set  $\sigma = ab - \eta a - \eta b$ . Then

$$\sigma = ((1 - \eta)\varphi_b(a) - \eta)b - \eta\alpha_b(a) = ((1 - \eta)\varphi_a(b) - \eta)a - \eta\alpha_a(b).$$

In particular  $\sigma$  is fixed by both  $\tau(a)$  and  $\tau(b)$ .

PROOF. By symmetry, we need only prove one equality from each part.

(a) We have

$$a = \varphi_b(a)b + \alpha_b(a) + \gamma_b(a)$$

hence

$$a^{\tau(b)} = \varphi_b(a)b + \alpha_b(a) - \gamma_b(a),$$

so  $a - a^{\tau(b)} = 2\gamma_b(a)$ .

(b) Next

$$\begin{aligned} a + a^{\tau(b)} &= (\varphi_b(a)b + \alpha_b(a) + \gamma_b(a)) + (\varphi_b(a)b + \alpha_b(a) - \gamma_b(a)) \\ &= 2(\varphi_b(a)b + \alpha_b(a)). \end{aligned}$$

(c) Furthermore

$$ab = (\varphi_b(a)b + \alpha_b(a) + \gamma_b(a))b = \varphi_b(a)b + \gamma_b(a)b = \varphi_b(a)b + \eta\gamma_b(a).$$

(d) Finally

$$\begin{aligned} \sigma &= ab - \eta a - \eta b \\ &= (\varphi_b(a)b + \eta\gamma_b(a)) - \eta(\varphi_b(a)b + \alpha_b(a) + \gamma_b(a)) - \eta b \\ &= ((1 - \eta)\varphi_b(a) - \eta)b - \eta\alpha_b(a), \end{aligned}$$

which is in  $A_1(b) + A_0(b)$  and so is fixed by  $\tau(b)$ . □

**(4.2).** LEMMA.

(a)  $a\sigma = ((1 - \eta)\varphi_a(b) - \eta)a.$

(b)  $b\sigma = ((1 - \eta)\varphi_b(a) - \eta)b.$

PROOF. We need only consider (b). From Lemma (4.1)

$$b\sigma = b(((1 - \eta)\varphi_b(a) - \eta)b - \eta\alpha_b(a)) = ((1 - \eta)\varphi_b(a) - \eta)b,$$

as claimed.  $\square$

**(4.3).** LEMMA. (Seress Lemma) *For  $p \in \mathcal{A}$ ,  $x \in A$ , and  $y \in A_1(p) + A_0(p)$*

$$p(xy) = (px)y.$$

PROOF. If  $z = \alpha p \in A_1(p)$ , then

$$p(xz) = p(x(\alpha p)) = (\alpha p)(xp) = (xp)(\alpha p) = (px)z;$$

so by linearity we may assume  $y \in A_0(p)$ .

Also by linearity we may assume that  $x \in A_\lambda(p)$ . In particular if  $x = \beta p \in A_1(p)$ , then

$$p(xy) = p(\beta py) = 0 = \beta py = (\beta pp)y = (px)y.$$

Finally if  $x \in A_\lambda(p)$  for  $\lambda \neq 1$ , then by the fusion rules  $xy \in A_\lambda$  also; so

$$p(xy) = \lambda(xy) = (\lambda x)y = (px)y. \quad \square$$

**(4.4).** LEMMA.  $(ab)\sigma = \pi ab$  for  $\pi = (1 - \eta)\varphi - \eta \in \mathbb{F}$  with  $\varphi = \varphi_a(b) = \varphi_b(a)$ .

PROOF. If  $ab = 0$ , then certainly  $(ab)\sigma = 0 = \pi ab$ . In that case  $b \in A_0(a)$  and  $a \in A_0(b)$  so that also  $\varphi_a(b) = 0 = \varphi_b(a)$ . Thus we may assume that  $ab \neq 0$ .

As  $\sigma$  is fixed by  $\tau(a)$  and  $\tau(b)$ , we have

$$\sigma \in (A_1(a) + A_0(a)) \cap (A_1(b) + A_0(b)).$$

Therefore by the previous lemma

$$b(a\sigma) = (ba)\sigma = (ab)\sigma = a(b\sigma),$$

and so by Lemma (4.2)

$$\begin{aligned}
((1 - \eta)\varphi_a(b) - \eta)ab &= b(((1 - \eta)\varphi_a(b) - \eta)a) \\
&= b(a\sigma) = a(b\sigma) \\
&= a(((1 - \eta)\varphi_b(a) - \eta)b) \\
&= ((1 - \eta)\varphi_b(a) - \eta)ab.
\end{aligned}$$

As  $ab \neq 0$

$$(1 - \eta)\varphi_a(b) - \eta = (1 - \eta)\varphi_b(a) - \eta = \pi$$

and especially  $\varphi_a(b) = \varphi_b(a)$  since  $\eta \neq 1$ . □

**(4.5).** LEMMA.  $\sigma^2 = \pi\sigma$ .

PROOF. By Lemmas (4.2) and (4.4)

$$\sigma^2 = (ab - \eta a - \eta b)\sigma = \pi ab - \eta\pi a - \eta\pi b = \pi\sigma. \quad \square$$

**(4.6).** PROPOSITION.  $A = \mathbb{F}a + \mathbb{F}b + \mathbb{F}\sigma$  with multiplication table:

$\cdot$	$a$	$b$	$\sigma$
$a$	$a$	$\eta a + \eta b + \sigma$	$\pi a$
$b$	$\eta a + \eta b + \sigma$	$b$	$\pi b$
$\sigma$	$\pi a$	$\pi b$	$\pi\sigma$

where  $\pi = (1 - \eta)\varphi - \eta$ .

PROOF. The multiplication table is immediate from the previous lemmas and our definitions of  $a$ ,  $b$ , and  $\sigma$ . As the span  $\mathbb{F}a + \mathbb{F}b + \mathbb{F}\sigma$  contains the generators of  $A$  and contains the pairwise products of its three spanning elements, it is all of  $A$ . □

We are not claiming in this proposition that  $a$ ,  $b$ , and  $\sigma$  are linearly independent. Indeed in algebras of type 1A and 2B this is certainly not the case.

**(4.7).** THEOREM. Choose parameters  $\eta (\neq 0, 1)$  and  $\varphi$  in  $\mathbb{F}$ . Let  $B = B(\eta, \varphi) = \mathbb{F}c \oplus \mathbb{F}d \oplus \mathbb{F}\rho$  be the commutative  $\mathbb{F}$ -algebra with multiplication table:

$\cdot$	$c$	$d$	$\rho$
$c$	$c$	$\eta c + \eta d + \rho$	$\pi c$
$d$	$\eta c + \eta d + \rho$	$d$	$\pi d$
$\rho$	$\pi c$	$\pi d$	$\pi \rho$

where  $\pi = (1 - \eta)\varphi - \eta$ .

- (a)  $B(\eta, \varphi)$  is a primitive axial algebra generated by axes  $\mathcal{B} = \{c, d\} = \{p, q\}$  with

$$B = B_1(p) \oplus B_0(p) \oplus B_\eta(p)$$

for  $B_1(p) = \mathbb{F}p$ ,  $B_0(p) = \mathbb{F}(\pi p - \rho)$ , and  $B_\eta(p) = \mathbb{F}((\eta - \varphi)p + \eta q + \rho)$ .

- (b)  $B(\eta, \varphi)$  satisfies the Seress condition:

$$B_0(c)B_\lambda(c) \subseteq B_\lambda(c) \text{ for } \lambda \neq 1.$$

- (c) For the generating set of axes  $\mathcal{B}$  we have the fusion table:

$\star$	1	0	$\eta$
1	1	$\emptyset$	$\eta$
0	$\emptyset$	0	$\eta$
$\eta$	$\eta$	$\eta$	1, 0, $\eta$

- (d)  $B(\eta, \varphi)$  is simple except in the following cases:

- (i)  $\varphi = \frac{\eta}{1-\eta}$  and  $\pi = 0$  where  $B_0(c) = B_0(d) = \mathbb{F}\rho$  is an ideal;
- (ii)  $\varphi = 0$  and  $\pi = -\eta$  where  $B_\eta(c) = B_\eta(d) = \mathbb{F}(\eta c + \eta d + \rho)$  is an ideal as are

$$B_0(c) \oplus B_\eta(c) = \mathbb{F}d \oplus \mathbb{F}(\eta c + \eta d + \rho) = B_1(d) + B_\eta(d)$$

and

$$B_0(d) \oplus B_\eta(d) = \mathbb{F}c \oplus \mathbb{F}(\eta c + \eta d + \rho) = B_1(c) \oplus B_\eta(c);$$

(iii)  $\varphi = 1$  and  $\pi = 1 - 2\eta$  where

$$B_0(c) \oplus B_\eta(c) = B_0(d) \oplus B_\eta(d)$$

is an ideal.

PROOF. (a) Certainly  $p \in B_1(p)$ . Also

$$p(\pi p - \rho) = \pi p^2 - p\rho = \pi p - \pi p = 0,$$

so  $\pi p - \rho \in B_0(p)$ . Next

$$\begin{aligned} p((\eta - \varphi)p + \eta q + \rho) &= (\eta - \varphi)p^2 + \eta pq + p\rho \\ &= (\eta - \varphi)p + \eta(\eta p + \eta q + \rho) + \pi p \\ &= (\eta - \varphi + \eta^2 + \pi)p + \eta^2 q + \eta\rho \\ &= (\eta - \varphi + \eta^2 + \varphi - \eta\varphi - \eta)p + \eta^2 q + \eta\rho \\ &= \eta((\eta - \varphi)p + \eta q + \rho), \end{aligned}$$

hence  $(\eta - \varphi)p + \eta q + \rho \in B_\eta(p)$ . As  $\eta \neq 0$ , the three vectors  $p$ ,  $\pi p - \rho$ , and  $(\eta - \varphi)p + \eta q + \rho$  are linearly independent in  $B$  of dimension 3.

(b) The Seress Condition speaks to the entries of the fusion table corresponding to  $B_0(p)B_\lambda(p)$  with  $\lambda \in \{0, \eta\}$ . Here  $B_0(p)$  is spanned by  $\pi p - \rho$ . But each  $B_\lambda(p)$  is an eigenspace for  $p$  by definition, while  $\rho$  acts on  $B$  as scalar multiplication by  $\pi$ . Therefore in all cases  $B_0(p)B_\lambda(p) \subseteq B_\lambda(p)$ .

(c) The fusion table summarizes parts of (a) and (b).

(d) As the adjoint eigenvalues 1, 0, and  $\eta$  are distinct, any ideal is a direct sum of certain of the 1-dimensional eigenspaces  $B_1(c)$ ,  $B_0(c)$ , and  $B_\eta(c)$  and equally well of  $B_1(d)$ ,  $B_0(d)$ , and  $B_\eta(d)$ .

No ideal of dimension 1 can contain  $c$  or  $d$ , since the quotient of dimension 2 would be generated by a single idempotent. Therefore a 1-dimensional ideal is one of  $B_0(c)$  or  $B_\eta(c)$  that is simultaneously equal to one of  $B_0(d)$  or  $B_\eta(d)$ .

First consider an ideal  $B_0(c) = \mathbb{F}(\pi c - \rho)$ . It must also contain

$$\begin{aligned} (\pi c - \rho)d &= \pi dc - d\rho \\ &= \pi(\eta c + \eta d + \rho) - \pi d \\ &= \pi(\eta c + (\eta - 1)d + \rho), \end{aligned}$$

As  $\eta \neq 1$ , the element  $\pi(\eta c + (\eta - 1)d + \rho)$  is a scalar multiple of  $\pi c - \rho$  in  $B(\eta, \varphi)$  if and only if  $\pi = 0$  hence  $\varphi = \frac{\eta}{1-\eta}$ . The corresponding ideal of



dimension 1 is  $\mathbb{F}\rho$ , which is  $B_0(d)$  as well. That is,  $B_0(c)$  is an ideal if and only if  $B_0(d)$  is an ideal, in which case both are  $\mathbb{F}\rho$ .

By the above,  $B_\eta(c)$  is an ideal if and only if it equals  $B_\eta(d)$ . The generators are, respectively,  $(\eta - \varphi)c + \eta d + \rho$  and  $\eta c + (\eta - \varphi)d + \rho$ , equal precisely when  $\varphi = 0$ . The corresponding ideal of dimension 1 is then

$$B_\eta(c) = \mathbb{F}(\eta c + \eta d + \rho) = B_\eta(d).$$

An ideal of dimension 2 in  $B(\eta, \varphi)$  not containing  $c$  must be

$$I(\eta, \varphi) = B_0(c) \oplus B_\eta(c) = \mathbb{F}(\pi c - \rho) \oplus \mathbb{F}((\eta - \varphi)c + \eta d + \rho).$$

The element  $d$  must map to an idempotent  $\bar{d}$  in  $\bar{B} = B(\eta, \varphi)/I(\eta, \varphi)$ , which is isomorphic to axial algebra  $\mathbb{F}$  of type 1A. Therefore  $\bar{d}$  is either  $\bar{c}$ , so that  $c - d \in I(\eta, \varphi)$ , or  $\bar{0}$ , which is to say  $d \in I(\eta, \varphi)$ . On the other hand,

$$d = \varphi c + \eta^{-1}((\pi c - \rho) + ((\eta - \varphi)c + \eta d + \rho)) \in \varphi c + I(\eta, \varphi);$$

so the two cases lead, respectively, to  $\varphi = 1$  and  $\varphi = 0$ .

If  $\varphi = 1$ , then  $\pi = 1 - 2\eta$ . Here the ideal  $I(\eta, \varphi)$  is

$$\begin{aligned} B_0(c) \oplus B_\eta(c) &= \mathbb{F}(\pi c - \rho) \oplus \mathbb{F}((\eta - 1)c + \eta d + \rho) \\ &= \mathbb{F}((1 - 2\eta)c - \rho) \oplus \mathbb{F}((\eta - 1)c + \eta d + \rho) \\ &= \mathbb{F}(c - d) \oplus \mathbb{F}(-\pi d + \rho) \\ &= B_0(d) \oplus B_\eta(d). \end{aligned}$$

Next suppose  $\varphi = 0$ . A 2-dimensional ideal that contains  $d$  but not  $c$  is thus

$$B_0(c) \oplus B_\eta(c) = \mathbb{F}d \oplus \mathbb{F}(\eta c + \eta d + \rho) = B_1(d) + B_\eta(d).$$

while by symmetry a 2-dimensional ideal containing  $c$  but not  $d$  is

$$B_0(d) \oplus B_\eta(d) = \mathbb{F}c \oplus \mathbb{F}(\eta c + \eta d + \rho) = B_1(c) + B_\eta(c). \quad \square$$

**(4.8). PROPOSITION.** *Let  $\bar{B}$  be a quotient of  $B = B(\eta, \varphi)$ . Then  $\bar{B}$  is an axial algebra of Jordan type  $\eta$  if and only if we have one of:*

- (1)  $\bar{B}$  is associative and isomorphic to  $\mathbb{F}$  of type 1A or  $\mathbb{F} \oplus \mathbb{F}$  of type 2B;
- (2)  $\varphi = \frac{\eta}{2}$ ;
- (3)  $\eta = \frac{1}{2}$ .

*In all these cases  $\bar{B}$  is spanned by  $\bar{c}$ ,  $\bar{d}$ , and either of  $\bar{c}^{\tau(\bar{d})}$  or  $\bar{d}^{\tau(\bar{c})}$ .*

PROOF. By the previous theorem,  $\bar{B}$  is of Jordan type  $\eta$  precisely when

$$\bar{B}_\eta(\bar{p})\bar{B}_\eta(\bar{p}) \subseteq \bar{B}_1(\bar{p}) \oplus \bar{B}_0(\bar{p}).$$

where  $\{\bar{p}, \bar{q}\} = \{\bar{c}, \bar{d}\}$ .

In  $B = B(\eta, \varphi)$  we have  $B_\eta(p) = \mathbb{F}((\eta - \varphi)p + \eta q + \rho)$ . Thus  $\bar{B}_\eta(\bar{p})$  is spanned by  $(\eta - \varphi)\bar{p} + \eta\bar{q} + \bar{\rho}$ . To check fusion containment we calculate

$$\begin{aligned} & ((\eta - \varphi)\bar{p} + \eta\bar{q} + \bar{\rho})((\eta - \varphi)\bar{p} + \eta\bar{q} + \bar{\rho}) \\ &= (\eta - \varphi)^2\bar{p}^2 + \eta^2\bar{q}^2 + \bar{\rho}^2 + 2\eta\bar{q}\bar{\rho} + 2(\eta - \varphi)\bar{p}\bar{\rho} + 2(\eta - \varphi)\eta\bar{p}\bar{q} \\ &= (\eta - \varphi)^2\bar{p} + \eta^2\bar{q} + \pi\bar{\rho} + 2\pi\eta\bar{q} + 2\pi(\eta - \varphi)\bar{p} + 2(\eta - \varphi)\eta(\eta\bar{p} + \eta\bar{q} + \bar{\rho}) \\ &= ((\eta - \varphi)^2 + 2\pi(\eta - \varphi) + 2(\eta - \varphi)\eta^2)\bar{p} + (\eta^2 + 2\pi\eta + 2(\eta - \varphi)\eta^2)\bar{q} \\ &\quad + (\pi + 2(\eta - \varphi)\eta)\bar{\rho}. \end{aligned}$$

Therefore  $\bar{B}$  is an axial algebra of Jordan type  $\eta$  if and only if the subalgebra  $\bar{B}_1(\bar{p}) \oplus \bar{B}_0(\bar{p}) = \mathbb{F}\bar{p} \oplus \mathbb{F}(\pi\bar{p} - \bar{\rho}) = \mathbb{F}\bar{p} \oplus \mathbb{F}\bar{\rho}$  contains the element

$$\begin{aligned} (\eta^2 + 2\pi\eta + 2(\eta - \varphi)\eta^2)\bar{q} &= (\eta^2 + 2((1 - \eta)\varphi - \eta)\eta + 2(\eta - \varphi)\eta^2)\bar{q} \\ &= (\eta^2 + 2\varphi\eta - 2\varphi\eta^2 - 2\eta^2 + 2\eta^3 - 2\varphi\eta^2)\bar{q} \\ &= \eta(2\eta^2 + (-1 - 4\varphi)\eta + 2\varphi)\bar{q} \\ &= \eta(2\eta - 1)(\eta - 2\varphi)\bar{q}. \end{aligned}$$

As  $\mathbb{F}$  and  $\mathbb{F} \oplus \mathbb{F}$  have Jordan type  $\eta$  for all  $\eta$ , this immediately gives us the converse part of the proposition, the claim about spanning following from Lemma (4.1)(c).

Now assume that  $\bar{B}$  does have Jordan type  $\eta$  but  $\varphi \neq \frac{\eta}{2}$  and  $\eta \neq \frac{1}{2}$ . Then  $\eta(\eta - 2\varphi)(2\eta - 1)$  is nonzero in  $\mathbb{F}$ , and so

$$\bar{d} \in \mathbb{F}\bar{c} \oplus B_0(\bar{c}) = \bar{B} \quad \text{and} \quad \bar{c} \in \mathbb{F}\bar{d} \oplus B_0(\bar{d}) = \bar{B}.$$

By Corollary (2.9) the axial algebra generated by  $\bar{c}$  and  $\bar{d}$  is associative and isomorphic to  $\mathbb{F}$  or to  $\mathbb{F} \oplus \mathbb{F}$ , as desired.  $\square$

PROOF OF THEOREM (1.1).

By Propositions (4.6) and (4.8) the algebras  $A$  of the theorem are  $\mathbb{F}$ ,  $\mathbb{F} \oplus \mathbb{F}$ , and the quotients of  $B(\eta, \frac{\eta}{2})$  and  $B(\frac{1}{2}, \varphi)$ . We claim:

- (i)  $B(\eta, \frac{\eta}{2})$  is isomorphic to  $3C(\eta)$ .
- (ii)  $B(\frac{1}{2}, \varphi)$  is isomorphic to  $Cl^J(\mathbb{F}^2, b_\delta)$  for  $\varphi \neq 1$ ,  $\delta = 4\varphi - 2 \neq 2$ , and  $B(\frac{1}{2}, 1)$  is isomorphic to  $Cl^{00}(\mathbb{F}^2, b_2)$ .

(i) As  $B(\eta, \frac{\eta}{2})$  and  $3C(\eta)$  both have dimension 3, it is enough to note that the axial algebra  $3C(\eta)$  has Jordan type  $\eta$ . But this was shown in (3.3).

(ii) By (3.5) and (3.6) the 3-dimensional algebras  $\text{Cl}^J(\mathbb{F}^2, b_\delta)$  ( $\delta \neq 2$ ) and  $\text{Cl}^{00}(\mathbb{F}^2, b_2)$  ( $\delta = 2$ ) are quotients of the 3-dimensional algebras  $B(\frac{1}{2}, \varphi)$  for  $\pi = \frac{1}{8}(\delta - 2)$ . As  $\pi = (1 - \eta)\varphi - \eta = \frac{1}{2}\varphi - \frac{1}{2}$ , this gives  $\delta = 4\varphi - 2$ .

The proper quotients of  $B(\eta, \varphi)$  are detailed in Theorem (4.7). A quotient of dimension 1 must have type 1A and need not be discussed further. A quotient of dimension 2 with  $\varphi = 0$  has  $\bar{B}_\eta(p) = 0$  and so is of type 2B. Therefore we only need consider quotients of dimension 2 with  $\varphi = \frac{\eta}{1-\eta}$ . This leads to  $\bar{B}(-1, -\frac{1}{2})$ , which is isomorphic to  $3C(-1)^*$  by (3.3), and  $\bar{B}(\frac{1}{2}, 1)$ , which is isomorphic to  $\text{Cl}^0(\mathbb{F}^2, b_2)$  by (3.6).  $\square$

**(4.9). REMARKS.** (1) *For an algebra of dimension 3 to appear under both (i) and (ii) we must have  $\eta = \frac{1}{2}$  and  $\varphi = \frac{1}{4}$  so that  $\delta = 4\varphi - 2 = -1$ . Thus the only dimension 3 algebra to occur in both is  $3C(\frac{1}{2})$  which is  $\text{Cl}^J(\mathbb{F}^2, b_{-1})$  when the characteristic is not three and  $\text{Cl}^{00}(\mathbb{F}^2, b_2)$  in characteristic three, which in turn leads to  $3C(-1)^* = \text{Cl}^0(\mathbb{F}^2, b_2)$  in characteristic three.*

(2) *The previous remark does not completely solve the isomorphism problem for the conclusions to Theorem (1.1), since that theorem actually provides a classification up to isomorphism of axial algebras of Jordan type  $\eta$  equipped with two marked generators. As mentioned under (3.5), isomorphic 2-generated algebras  $A$  and  $A'$  can nevertheless give rise to nonisomorphic marked algebras  $(A, a, b)$  and  $(A', a', b')$ . Section 4 of [HRS13] discusses categories of marked algebras in detail.*

Now consider a primitive axial algebra  $A$  of Jordan type  $\eta$  with generating axis set  $\mathcal{A}$  of arbitrary size (not necessarily two). Recall that  $\bar{\mathcal{A}}$  is the smallest set of axes with the properties:

- (i)  $\mathcal{A} \subseteq \bar{\mathcal{A}}$ .
- (ii) If  $p \in \bar{\mathcal{A}}$  and  $t$  is the Miyamoto involution associated with  $p$ , then  $\bar{\mathcal{A}}^t \subseteq \bar{\mathcal{A}}$ .

Corollary (1.2) states that the algebra  $A$  is the  $\mathbb{F}$ -space spanned by the axes of  $\bar{\mathcal{A}}$ .

**PROOF OF COROLLARY (1.2).**

The algebra  $A$  is spanned as  $\mathbb{F}$ -space by the multiplicative submagma generated by  $\mathcal{A}$ . By Theorem (1.1) and Proposition (4.8) every product of two members of  $\mathcal{A}$ , and indeed any two members of  $\bar{\mathcal{A}}$ , is in the  $\mathbb{F}$ -span of

$\bar{\mathcal{A}}$ . Therefore the span of  $\bar{\mathcal{A}}$  is closed under multiplication and contains the generators  $\mathcal{A}$ ; it is equal to  $A$ .  $\square$

## 5 Automorphisms of axial algebras of Jordan type

In this section we focus on the automorphism groups of axial algebras of Jordan type  $\eta$  that are generated by Miyamoto involutions. Especially we examine dihedral subgroups generated by two Miyamoto involutions.

The following observation will be used frequently without mention.

**(5.1). LEMMA.** *If  $t$  is an automorphism of  $A$  and  $m$  is an axis, then  $m^t$  is an axis with  $\tau(m)^t = \tau(m^t)$ .*  $\square$

We first consider axes  $a$  for which  $\tau(a) = 1$ .

**(5.2). LEMMA.** *Let  $A$  be an axial algebra of Jordan type  $\eta$  over the field  $\mathbb{F}$  that is generated by the set  $\mathcal{A}$  of axes. Write  $\mathcal{A}$  as the disjoint union of  $\mathcal{A}^1$  and  $\mathcal{A}^\eta$ , where  $a \in \mathcal{A}^1$  if and only if  $\tau(a) = 1$  and  $a \in \mathcal{A}^\eta$  if and only if  $\tau(a)$  has order 2. Then  $A = (\bigoplus_{a \in \mathcal{A}^1} \mathbb{F}a) \oplus A^\eta$  where  $A^\eta$  is the axial algebra of Jordan type  $\eta$  generated by  $\mathcal{A}^\eta$ .*

**PROOF.** By Proposition (2.1), for an algebra  $A$  of Jordan type we have  $\tau(a) = 1$  precisely when  $A = \mathbb{F}a \oplus A_0(a)$ . In this case by Proposition (2.8) the subalgebra  $A_0(a)$  contains all the axes of  $\mathcal{A}$  except  $a$ . Especially  $\bigcap_{a \in \mathcal{A}^1} A_0(a)$  is the subalgebra  $A^\eta$  generated by  $\mathcal{A}^\eta$  and then  $A = (\bigoplus_{a \in \mathcal{A}^1} \mathbb{F}a) \oplus A^\eta$ , as claimed.  $\square$

Thus we are able to reduce to the case where all  $\tau(m)$  have order 2 and are Miyamoto involutions.

For  $\eta = \frac{1}{2}$ , the examples coming from Clifford algebras (and described in (3.5)) have  $\tau(e_0^+)\tau(e_1^+) = t_{v_0}t_{v_1}$ , the product of two orthogonal reflections on the space  $\mathbb{F}^2$ . All orders are possible (although restrictions on the field  $\mathbb{F}$  and the form  $b$  would lead to order restrictions). Especially, for infinite  $\mathbb{F}$  it is possible for the two Miyamoto involutions to generate an infinite dihedral group so that finite  $\mathcal{A}$  (of size two) generates an algebra of finite dimension but with  $\bar{\mathcal{A}}$  infinite. We will have little more to say regarding the case  $\eta = \frac{1}{2}$  in this section.

For  $\eta \neq \frac{1}{2}$ , the situation is much different. Theorem (1.1) tells us that the possibilities for the dihedral group generated by two Miyamoto involutions are extremely limited.

The remainder of this section is focused on proving a more precise version of Theorem (1.3):

**(5.3). THEOREM.** *Let  $A$  be an axial algebra of Jordan type  $\eta \neq \frac{1}{2}$  over a field of characteristic not two that is generated by the set  $\mathcal{A}$  of axes.*

*Write  $\mathcal{A}$  as the disjoint union of  $\mathcal{A}^1$  and  $\mathcal{A}^\eta$ , where  $a \in \mathcal{A}^1$  if and only if  $\tau(a) = 1$  and  $a \in \mathcal{A}^\eta$  if and only if  $\tau(a)$  has order 2.*

- (a)  $A = \left(\bigoplus_{a \in \mathcal{A}^1} \mathbb{F}a\right) \oplus A^\eta$  where  $A^\eta$  is the axial algebra of Jordan type  $\eta$  generated by  $\mathcal{A}^\eta$ .
- (b) The map  $a \mapsto \tau(a)$  is a bijection of  $\bar{\mathcal{A}}^\eta$  with the corresponding set  $D$  of Miyamoto involutions, and  $D$  is a normal set of 3-transpositions in the subgroup  $\langle D \rangle$  of the automorphism group of  $A$  and  $A^\eta$ .

A normal set  $D$  of elements of order 2 in the group  $G$  is said to consist of 3-transpositions provided, for each pair  $d, e \in D$  the order of the product  $de$  is 1, 2, or 3. Equivalently, we must (respectively) have one of  $d = e$ ,  $de = ed \neq 1$ , or  $\langle d, e \rangle \simeq \text{Sym}(3)$ .

The elements of  $D$  are called *transpositions* as the motivating examples are the transpositions of any symmetric group. A group generated by a normal set of 3-transpositions is called a *3-transposition group*. These have been studied extensively since their introduction by Fischer [Fi71]. Fischer's work and its successors, especially [CuHa95], effectively classify all 3-transposition groups.

**(5.4). PROPOSITION.** *Let  $A$  be an axial algebra of Jordan type  $\eta$ . Let  $a$  and  $b$  be two axes of  $A$  with  $\tau(a)$  and  $\tau(b)$  the corresponding Miyamoto involutions, and let  $N$  be the subalgebra generated by  $a$  and  $b$ .*

- (a) *If  $N$  is of type 1A, then  $a = b$ ,  $\tau(a) = \tau(b)$ , and  $\tau(a)\tau(b) = 1$ .*
- (b) *If  $N$  is of type 2B, then  $\tau(a)\tau(b) = \tau(b)\tau(a)$  and  $(\tau(a)\tau(b))^2 = 1$ .*
- (c) *If  $N$  is of type  $3C(\eta)$  or type  $3C(-1)^*$ , then  $\tau(a)^{\tau(b)} = \tau(b)^{\tau(a)}$  and  $(\tau(a)\tau(b))^3 = 1$ .*

PROOF. We consider the cases in turn.

(a) Here  $N$  contains a single axis  $a = b$ , so  $\tau(a) = \tau(b)$ .

(b) By (3.2) we have  $b^{\tau(a)} = b$ , so  $\tau(b)^{\tau(a)} = \tau(b)$  and  $(\tau(a)\tau(b))^2 = \tau(b)^{\tau(a)}\tau(b) = 1$ .

(c) By (3.3) we have  $b^{\tau(a)} = c = a^{\tau(b)}$ . Therefore

$$\tau(b)^{\tau(a)} = \tau(c) = \tau(b)^{\tau(a)}$$

and

$$(\tau(a)\tau(b))^3 = \tau(b)^{\tau(a)}\tau(a)^{\tau(b)} = \tau(c)\tau(c) = 1. \quad \square$$

**(5.5). PROPOSITION.** *Let  $A$  be a primitive axial algebra of Jordan type  $\eta$  over a field of characteristic not two that is generated by the set  $\mathcal{A}$  of axes. Assume that every subalgebra generated by two elements of  $\mathcal{A}$  has type one of 1A, 2B,  $3C(\eta)$ , or  $3C(-1)^*$ . (By Theorem (1.1), this is the case for  $\eta \neq \frac{1}{2}$ .) If  $a, b \in \mathcal{A}$  with  $\tau(a) = \tau(b) \neq 1$  then  $a = b$ .*

PROOF. Let  $t = \tau(a) = \tau(b)$  and choose  $c \in \mathcal{A}$  with  $c^t \neq c$ , possible as  $t \neq 1$ . Let  $N_{a,c}$  be the subalgebra generated by  $a$  and  $c$ , and let  $N_{b,c}$  be the subalgebra generated by  $b$  and  $c$ . By hypothesis the types of  $N_{a,c}$  and  $N_{b,c}$  are  $3C(\eta)$  or possibly  $3C(-1)^*$  (when  $\eta = -1$ ). In particular,  $|\bar{\mathcal{A}} \cap N_{a,c}| = |\bar{\mathcal{A}} \cap N_{b,c}| = 3$  with  $\langle \tau(a), \tau(c) \rangle = \langle \tau(b), \tau(c) \rangle$  acting as the symmetric group of degree 3 on each. But then

$$a^{\tau(c)} = c^{\tau(a)} = c^t = c^{\tau(b)} = b^{\tau(c)},$$

hence  $a = b$ . □

By (3.5) when  $\eta = \frac{1}{2}$  distinct axes can have the same Miyamoto involution.

By Lemma (5.2) it is possible to have  $\tau(a) = \tau(b) = 1$  for distinct  $a$  and  $b$  in  $\mathcal{A}$ , but then the corresponding 1-dimensional subalgebras can be “subtracted out.”

PROOF OF THEOREM (5.3).

Part (a) follows directly from Lemma (5.2).

As  $\eta \neq \frac{1}{2}$  by assumption, Proposition (5.5) says that the map  $a \mapsto \tau(a)$  is a bijection of  $\bar{\mathcal{A}}^\eta$  and  $D = \{\tau(a) \mid a \in \bar{\mathcal{A}}^\eta\}$ . By definition  $(\bar{\mathcal{A}}^\eta)^t = \bar{\mathcal{A}}^\eta$  for each  $t \in D$ , so  $D^t = D$  is a normal subset in the subgroup  $\langle D \rangle$  of the automorphism groups of  $A$  and  $A^\eta$ . Theorem (1.1) tells us that any two elements  $a$  and  $b$  of  $\bar{\mathcal{A}}^\eta$  generate a subalgebra of type 1A, 2B,  $3C(\eta)$ , or  $3C(-1)^*$ . Then by Proposition (5.4) the corresponding product  $\tau(a)\tau(b)$  of two elements from  $D$  has order 1, 2, or 3. This gives (b). □

Theorem (5.3) includes Theorem (1.3). Corollary (1.4) then states that an axial algebra  $A$  of Jordan type  $\eta \neq \frac{1}{2}$  generated by a finite number of axes are finite dimensional.

PROOF OF COROLLARY (1.4).

Theorem (5.3) allows us to assume that  $\tau(a)$  has order 2 for every  $a \in \mathcal{A}$  and that there is a bijection between the Miyamoto involutions of the normal 3-transpositions set  $D$  and the elements of  $\bar{\mathcal{A}}$ .

By results from [CuHa95], every finitely generated 3-transposition group is finite. In particular, the group generated by the Miyamoto involutions

for finite  $\mathcal{A}$  is a finite group. As that subgroup contains the Miyamoto involutions for  $\bar{\mathcal{A}}$ , that set too is finite. The algebra is the  $\mathbb{F}$ -span of  $\bar{\mathcal{A}}$  by Corollary (1.2), so the algebra  $A$  is finite dimensional.  $\square$

In fact,  $|\bar{\mathcal{A}}|$  and so the dimension of  $A$  can be bounded by a function of  $|\mathcal{A}|$ ; see [HaSh].

## 6 Matsuo algebras and Fischer spaces

Versions of certain results from this section appeared originally in the unpublished work of Matsuo and Matsuo, [MaMa99] and [Ma03]. Related results also appear in [Re13].

A *partial triple system* or *partial linear space of order two*  $\Pi = (\mathcal{P}, \mathcal{L})$  is a set  $\mathcal{P}$ , called *points*, and a set  $\mathcal{L}$  of subsets of  $\mathcal{P}$ , called *lines*, such that:

- (i) every line contains exactly three points;
- (ii) two distinct points belong to at most one line.

If every pair of distinct points belongs to a unique line, then we have a *linear space of order two* or *Steiner triple system*.

For distinct points  $p, q$ , we write  $p \sim q$  if  $p$  and  $q$  are collinear and  $p \perp q$  if they are not. The set  $p^\perp$  is then the set of all points not collinear with  $p$ . Let  $\approx$  be the equivalence relation on  $\mathcal{P}$  generated by  $\sim$ . The  $\approx$ -equivalence classes are the *connected components* of  $\Pi$ .

A *subspace*  $(\mathcal{P}', \mathcal{L}')$  of  $(\mathcal{P}, \mathcal{L})$  is a subset  $\mathcal{P}'$  of  $\mathcal{P}$  with the property that whenever there is a line of  $\mathcal{L}$  intersecting  $\mathcal{P}'$  in at least two points, then the line is a subset of  $\mathcal{P}'$  and so a member of the line set  $\mathcal{L}' \subseteq \mathcal{L}$ . For instance, every line is itself a subspace. A *plane* is the subspace generated by two distinct, intersecting lines—the intersection of all subspaces containing the two lines. Each connected component of  $\Pi$  is a subspace, and  $\Pi$  is then the disjoint union of its connected components.

Of particular interest here are the *Fischer spaces*. If  $D$  is a normal set of 3-transpositions in the group  $G$ , then the associated Fischer space  $\Pi$  is (up to isomorphism) the partial triple system having point set  $D$  and line set consisting of the triples of points (transpositions) from the various subgroups  $\text{Sym}(3)$  generated by two transpositions.

For each  $p \in \mathcal{P}$  the associated transposition  $t(p)$  of  $D$  is an automorphism of  $\Pi$  that fixes  $p$  and each  $q$  of  $p^\perp$  while switching  $r$  and  $s$  whenever  $\{p, r, s\}$  is a line. In particular the subspace fixed by  $t(p)$  is the disjoint union of  $\{p\}$  and the subspace  $p^\perp$ .

Fischer spaces have a well known characterization due to Buekenhout:

**(6.1). PROPOSITION.** *A partial triple system is a Fischer space if and only if the only isomorphism types allowed for its planes are the dual affine plane of order two and the affine plane of order three.*  $\square$

Let  $\Pi = (\mathcal{P}, \mathcal{L})$  be a partial triple system. Choose a constant  $\delta$  in the field  $\mathbb{F}$ . The associated *Matsuo algebra*  $M(\Pi, \delta, \mathbb{F})$  over the field  $\mathbb{F}$  is the  $\mathbb{F}$ -space  $\bigoplus_{p \in \mathcal{P}} \mathbb{F}a_p$  with multiplication provided by:

- (i) For  $p \in \mathcal{P}$  we have  $a_p^2 = a_p$ .
- (ii) For distinct  $p, q \in \mathcal{P}$  with  $p$  and  $q$  not collinear,  $a_p a_q = 0$ .
- (iii) For distinct  $p, r \in \mathcal{P}$  with  $\{p, r, s\}$  a line,  $a_p a_r = \delta(a_p + a_r - a_s)$ .

For each line  $\{p, r, s\}$  set

$$a_{prs} = \eta a_p - a_r - a_s \quad \text{and} \quad g_{prs} = a_r - a_s.$$

**(6.2). THEOREM.** *The Matsuo algebra  $M(\Pi, \frac{\eta}{2}, \mathbb{F})$  is a primitive axial algebra for the basis  $\mathcal{A} = \{a_p \mid p \in \mathcal{P}\}$  of axes with eigenvalue set  $\{1, 0, \eta\}$ . It is the direct sum of its ideals  $M(\Pi^{(i)}, \frac{\eta}{2}, \mathbb{F})$ , where the  $\Pi^{(i)}$ ,  $i \in I$ , are the connected components of  $\Pi$ .*

PROOF. For each  $p \in \mathcal{P}$  and each line  $\{p, r, s\}$  the pair  $\{a_r, a_s\}$  from the canonical basis  $\mathcal{A}$  can be replaced by  $\{a_{prs}, g_{prs}\}$ . (While  $a_{prs} = a_{psr}$ , we choose only one of  $g_{prs}$  and  $g_{psr} = -g_{prs}$ .) Then by (3.3) we have

$$\begin{aligned} M_1(a_p) &= \mathbb{F}a_p \\ M_0(a_p) &= \bigoplus_{q \in p^\perp} \mathbb{F}a_q \oplus \bigoplus_{\{p, r, s\} \in \mathcal{L}} \mathbb{F}a_{prs} \\ M_\eta(a_p) &= \bigoplus_{\{p, r, s\} \in \mathcal{L}} \mathbb{F}g_{prs}. \end{aligned}$$

In particular  $M$  has axial basis  $\mathcal{A} = \{a_p \mid p \in \mathcal{P}\}$  with eigenvalue set  $\{1, 0, \eta\}$ , as claimed.

If  $p$  and  $q$  are points in different connected components of  $\Pi$ , then  $a_p a_q = 0$ . Therefore each  $\mathcal{A}^{(i)} = \{a_p \mid p \in \Pi^{(i)}\}$  is the basis of a subalgebra  $M^{(i)}$  isomorphic to  $M(\Pi^{(i)}, \frac{\eta}{2}, \mathbb{F})$  and such that  $M^{(i)} M^{(j)} = 0$  for  $i \neq j$ . Especially each  $M^{(i)}$  is an ideal, and the algebra is the direct sum of these ideals.  $\square$

From Theorem (5.3) we have immediately



**(6.3). THEOREM.** For  $\eta \neq \frac{1}{2}$ , every primitive axial algebra of Jordan type  $\eta$  is isomorphic to  $(\bigoplus_{i \in I} \mathbb{F}) \oplus M$ , for some index set  $I$ , where  $M$  is a quotient of the Matsuo algebra  $M(\Pi, \frac{\eta}{2}, \mathbb{F})$  associated with the Fischer space  $\Pi$  of all Miyamoto involutions.  $\square$

The next theorem gives a construction that provides a converse but also works in the case  $\eta = \frac{1}{2}$ .

**(6.4). THEOREM.** Let  $\mathbb{F}$  be a field of characteristic not two and  $\eta \in \mathbb{F}$  with  $\eta \neq 0, 1$ . If  $\Pi = (\mathcal{P}, \mathcal{L})$  is a Fischer space, then the associated Matsuo algebra  $M = M(\Pi, \frac{\eta}{2}, \mathbb{F})$  is a primitive axial algebra of Jordan type  $\eta$  generated by its basis of axes  $\mathcal{A} = \{a_p \mid p \in \mathcal{P}\}$ . Each transposition  $t(p)$  acts as the Miyamoto involution  $\tau(a_p)$ .

PROOF. By Theorem (6.2) the algebra  $M$  is primitive and axial with the basis  $\mathcal{A}$  of axes for the eigenvalues  $\{1, 0, \eta\}$ .

By (3.3) (and as observed in the proof of Theorem (6.2))  $M_0(a_p)$  is spanned by the elements  $a_q$ , for  $q \in p^\perp$ , and  $a_{prs} = \eta a_p - a_r - a_s$ , for  $\{p, r, s\} \in \mathcal{L}$ , while  $M_\eta(a_p)$  is spanned by the elements  $g_{prs} = a_r - a_s$ , for  $\{p, r, s\} \in \mathcal{L}$ .

The transposition  $t(p)$  acts on the basis elements of  $\mathcal{A}$  via  $a_q^{t(p)} = a_{q^{t(p)}}$ . Thus, in view of the previous paragraph,  $t(p)$  induces on  $M$  the Miyamoto involution  $\tau(a_p)$  for the  $\mathbb{Z}/2\mathbb{Z}$ -grading  $M_+(a_p) = M_1(a_p) \cup M_0(a_p)$  and  $M_-(a_p) = M_\eta(a_p)$ .

By Lemma (2.10), it remains to check that each  $M_0(a_p)$  is a subalgebra—that it is closed under multiplication by its spanning elements. Thus there are three cases to consider.

(1)  $a_q a_v$  with  $q, v \in p^\perp$ .

Certainly  $a_q^2 = a_q$ . If  $q \in v^\perp$  then  $a_q a_v = 0 \in M_0(a_p)$ . If  $\{q, v, w\} \in \mathcal{L}$ , then  $a_q a_v = \frac{\eta}{2}(a_q + a_v - a_w)$ . Since  $p^\perp$  is a subspace containing  $q$  and  $v$ , it also contains  $w$ . Thus  $a_w$  and  $a_q a_v$  are both in  $M_0(a_p)$ .

(2)  $a_q a_{prs}$  with  $q \in p^\perp$  and  $\{p, r, s\} \in \mathcal{L}$ .

If  $r$  or  $s$  is in the subspace  $q^\perp$ , then the entire line  $\{p, r, s\}$  is in  $q^\perp$  and  $a_q a_{prs} = 0 \in M_0(a_p)$ .

Now assume that  $r, s \notin q^\perp$ . Then  $p, q, r, s$  all belong to the plane generated by  $\{p, r, s\}$  and the line through  $q$  and  $r$ . As  $q \in p^\perp$ , this plane must be dual affine of order two, and we may take its line set to be

$$\{p, r, s\}, \{p, t, u\}, \{q, r, t\}, \{q, s, u\}.$$

Thus

$$\begin{aligned}
a_q a_{prs} &= a_q (\eta a_p - a_r - a_s) \\
&= \eta a_p a_q - a_q a_r - a_q a_s \\
&= -\frac{\eta}{2} (a_q + a_r - a_t) - \frac{\eta}{2} (a_q + a_s - a_u) \\
&= -\eta a_q + \frac{\eta}{2} (-a_r - a_s) - \frac{\eta}{2} (-a_t - a_u) \\
&= -\eta a_q + \frac{\eta}{2} a_{prs} - \frac{\eta}{2} a_{ptu} \in M_0(a_p).
\end{aligned}$$

(3)  $a_{prs} a_{ptu}$  with  $\{p, r, s\}, \{p, t, u\} \in \mathcal{L}$ .

We have  $a_{prs}^2 \in M_0(a_p)$  by (3.3). For two distinct intersecting lines  $\{p, r, s\}$  and  $\{p, t, u\}$ , the calculation ultimately depends upon the type of the plane  $\Delta$  they generate. In either case

$$\begin{aligned}
a_{prs} a_{ptu} &= (\eta a_p - a_r - a_s)(\eta a_p - a_t - a_u) \\
&= (-a_r - a_s)(\eta a_p - a_t - a_u) \\
&= -\eta a_r a_p - \eta a_s a_p + a_r a_t + a_r a_u + a_s a_t + a_s a_u \\
&= -\frac{\eta^2}{2} (a_p + a_r - a_s) - \frac{\eta^2}{2} (a_p + a_s - a_r) \\
&\quad + a_r a_t + a_r a_u + a_s a_t + a_s a_u \\
&= -\eta^2 a_p + a_r a_t + a_r a_u + a_s a_t + a_s a_u.
\end{aligned}$$

If  $\Delta$  is dual affine of order two, then we may take its lines to be those of (2), so that  $a_r a_u = 0 = a_s a_t$  and

$$\begin{aligned}
a_{prs} a_{ptu} &= -\eta^2 a_p + a_r a_t + a_r a_u + a_s a_t + a_s a_u \\
&= -\eta^2 a_p + a_r a_t + a_s a_u \\
&= -\eta^2 a_p + \frac{\eta}{2} (a_r + a_t - a_q) + \frac{\eta}{2} (a_s + a_u - a_q) \\
&= -\eta a_q - \frac{\eta}{2} (2\eta a_p - a_r - a_t - a_s - a_u) \\
&= -\eta a_q - \frac{\eta}{2} (a_{prs} + a_{ptu}) \in M_0(a_p).
\end{aligned}$$

On the other hand, if  $\Delta$  is affine of order three, then we may take the two additional lines of  $\Delta$  on  $p$  to be  $\{p, w, x\}$  and  $\{p, y, z\}$ . We then have

$$\begin{aligned}
a_{prs} a_{ptu} &= -\eta^2 a_p + a_r a_t + a_r a_u + a_s a_t + a_s a_u \\
&= -\eta^2 a_p + \eta (a_r + a_s + a_t + a_u) - \frac{\eta}{2} (a_w + a_x + a_y + a_z) \\
&= -\frac{\eta}{2} (2\eta a_p - 2(a_r + a_s) - 2(a_t + a_u) + (a_w + a_x) + (a_y + a_z)) \\
&= -\frac{\eta}{2} (2a_{prs} + 2a_{ptu} - a_{pwx} - a_{pyz}) \in M_0(a_p). \quad \square
\end{aligned}$$

Theorem (6.4) gives all parts of Theorem (1.5) of the introduction except for the existence of an associative form, which is handled in Corollary (7.4) below.

We have a pleasant consequence of the work in this section:

**(6.5). THEOREM.** *The Matsuo algebra  $M(\Pi, \delta, \mathbb{F})$  is an axial algebra of Jordan type if and only if  $\Pi$  is a Fischer space.*

PROOF. The only possible Jordan type is  $\eta = 2\delta$ .

Theorem (6.4) gives the converse part of this theorem immediately. For  $\eta \neq \frac{1}{2}$  the rest follows from Theorem (6.3), but this difficult result is not necessary in proving the direct part for arbitrary  $\eta$ .

Suppose that the Matsuo algebra  $M(\Pi, \delta, \mathbb{F})$  is an axial algebra of Jordan type  $\eta = 2\delta$  presented using the partial triple system  $\Pi = (\mathcal{P}, \mathcal{L})$ . Then for each  $x \in \mathcal{P}$ , the Miyamoto involution  $\tau(a_x)$  permutes the generating set  $\mathcal{A}_{\mathcal{P}} = \{a_p \mid p \in \mathcal{P}\}$ , taking 2B subalgebras to 2B subalgebras and  $3C(\eta)$  algebras to  $3C(\eta)$  algebras. Therefore the induced permutation  $t(x)$  of  $\mathcal{P}$  given by  $a_p^{\tau(a_x)} = a_{p^{t(x)}}$  is an automorphism of the partial triple system  $\Pi$ . Indeed it is the unique *central* automorphism of  $\Pi$  with *center*  $x$ —that is, it fixes  $x$  and all points not collinear with  $x$  and, for each line  $\{x, y, z\}$  on  $x$ , it switches  $y$  and  $z$ .

It is well-known, and easy to check, that a triple system admits all possible central automorphisms if and only if the collection of central automorphisms is a normal set of 3-transpositions in  $\text{Aut}(\Pi)$  with  $\Pi$  as the corresponding Fischer space. Indeed, for any automorphism  $g$  we always have  $t(x)^g = t(x^g)$ . In particular if distinct  $x$  and  $y$  are not collinear, then

$$t(y)^{t(x)} = t(x)t(y)t(x) = t(y),$$

and  $(t(x)t(y))^2 = 1$ , while if  $\{x, y, z\}$  is a line

$$t(x)t(y)t(x) = t(y)^{t(x)} = t(z) = t(x)^{t(y)} = t(y)t(x)t(y)$$

and  $(t(x)t(y))^3 = t(z)^2 = 1$ . Therefore  $\Pi$  is a Fischer space.  $\square$

## 7 Frobenius axial algebras of Jordan type

The results from this section essentially appeared in [Ma03], the unpublished first version of the published [Ma05].

**(7.1).** LEMMA.

- (a) *The algebra 2B is a Frobenius algebra. The bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle$  on this algebra is associative if and only if  $\langle\langle b_0, b_1 \rangle\rangle = 0 = \langle\langle b_1, b_0 \rangle\rangle$ .*
- (b) *The algebra 3C( $\eta$ ) is a Frobenius algebra. An associative bilinear form on this algebra is a scalar multiple of the form given by*

$$\langle\langle c_i, c_i \rangle\rangle = 1, \quad \langle\langle c_i, c_j \rangle\rangle = \frac{\eta}{2} \text{ for } i \neq j.$$

PROOF. By Proposition (2.7)(a) an associative form on these axial algebras is symmetric.

(a) The given forms are clearly associative. Now consider an arbitrary associative form. Then for  $i \neq j$ ,

$$\langle\langle b_i, b_j \rangle\rangle = \langle\langle b_i b_i, b_j \rangle\rangle = \langle\langle b_i, b_i b_j \rangle\rangle = \langle\langle b_i, 0 \rangle\rangle = 0.$$

(b) For any bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle$  on 3C( $\eta$ ) we have, for  $\{i, j, k\} = \{0, 1, 2\}$ :

$$(i) \quad \langle\langle c_i, c_i c_i \rangle\rangle = \langle\langle c_i, c_i \rangle\rangle = \langle\langle c_i c_i, c_i \rangle\rangle ;$$

(ii)

$$\begin{aligned} \langle\langle c_i, c_i c_j \rangle\rangle &= \frac{\eta}{2} \langle\langle c_i, c_i + c_j - c_k \rangle\rangle \\ &= \frac{\eta}{2} (\langle\langle c_i, c_i \rangle\rangle + \langle\langle c_i, c_j \rangle\rangle - \langle\langle c_i, c_k \rangle\rangle) ; \end{aligned}$$

(iii)

$$\begin{aligned} \langle\langle c_i, c_j c_k \rangle\rangle - \langle\langle c_i c_j, c_k \rangle\rangle &= \frac{\eta}{2} \langle\langle c_i, c_j + c_k - c_i \rangle\rangle - \frac{\eta}{2} \langle\langle c_i + c_j - c_k, c_k \rangle\rangle \\ &= \frac{\eta}{2} (\langle\langle c_i, c_j \rangle\rangle - \langle\langle c_j, c_k \rangle\rangle - \langle\langle c_i, c_i \rangle\rangle + \langle\langle c_k, c_k \rangle\rangle) . \end{aligned}$$

Therefore the form is associative if and only if the righthand side of (ii) is always equal to  $\langle\langle c_i, c_j \rangle\rangle$  and the righthand side of (iii) is always equal to 0. In particular, the given values do lead to an associative form (as promised earlier under (3.3)).

Now assume that the form is associative. By (3.3),  $\mathbb{F}(c_j - c_k)$  is the  $\eta$ -eigenspace for  $\text{ad}_{c_i}$ ; so by Proposition (2.7) always  $\langle\langle c_i, c_j - c_k \rangle\rangle = 0$ . Therefore

$$\langle\langle c_0, c_1 \rangle\rangle = \langle\langle c_0, c_2 \rangle\rangle = \langle\langle c_1, c_2 \rangle\rangle = \kappa ,$$

for some constant  $\kappa$ . Then as the righthand side of (iii) is always 0,

$$\langle\langle c_0, c_0 \rangle\rangle = \langle\langle c_1, c_1 \rangle\rangle = \langle\langle c_2, c_2 \rangle\rangle = k$$

is constant as well.

Finally as the form is associative, (ii) becomes

$$\kappa = \langle\langle c_i, c_j \rangle\rangle = \langle\langle c_i, c_i c_j \rangle\rangle = \frac{\eta}{2} (\langle\langle c_i, c_i \rangle\rangle + \langle\langle c_i, c_j \rangle\rangle - \langle\langle c_i, c_k \rangle\rangle) = \frac{\eta}{2} k.$$

The constant  $k$  determines the associative form up to a scalar multiple, as claimed, the given values corresponding to  $k = 1$ .  $\square$

**(7.2). THEOREM.** *Let  $\Pi = (\mathcal{P}, \mathcal{L})$  be a partial triple system. The Matsuo algebra  $M(\Pi, \frac{\eta}{2}, \mathbb{F})$  admits a nonzero associative form if and only if*

- (i) *for each  $x \in \mathcal{P}$ , the subset  $x^\perp$  is a subspace of  $\Pi$ , and*
- (ii) *if  $\{x, y, z\}$  and  $\ell = \{x, v, w\}$  are lines of  $\mathcal{L}$ , then  $\ell \cap y^\perp = \emptyset$  if and only if  $\ell \cap z^\perp = \emptyset$ .*

*When  $\Pi$  is connected, such a form is a scalar multiple of the form given by, for distinct  $p, q \in \mathcal{P}$ ,*

$$\langle\langle a_p, a_p \rangle\rangle = 1; \langle\langle a_p, a_q \rangle\rangle = 0 \text{ if } q \in p^\perp; \langle\langle a_p, a_q \rangle\rangle = \frac{\eta}{2} \text{ if } q \notin p^\perp.$$

PROOF. By the previous lemma, under any associative form the ideals corresponding to the distinct connected components of  $\Pi$  are perpendicular. Furthermore, the form when restricted to a specific component  $\Pi_i$  can only be a scalar multiple of the given form.

It remains to prove that this does give an associative form for the (connected) space  $\Pi (= \Pi_i)$  if and only if the two conditions (i) and (ii) are satisfied.

By linearity it suffices to prove

$$\langle\langle a_r a_p, a_t \rangle\rangle = \langle\langle a_r, a_p a_t \rangle\rangle$$

for all  $r, p, t \in \mathcal{P}$ , where by Lemma (7.1) we may assume that  $r, p, t$  are distinct and do not lie together in a line of  $\mathcal{L}$ .

If  $r, t \in p^\perp$ , then

$$\langle\langle a_r a_p, a_t \rangle\rangle = \langle\langle 0, a_t \rangle\rangle = 0 = \langle\langle a_r, 0 \rangle\rangle = \langle\langle a_r, a_p a_t \rangle\rangle,$$

as desired. Therefore we may also assume that  $r \sim p$ . Let  $\{p, r, s\}$  be the line on  $r$  and  $p$ .

CLAIM.  $\langle\langle a_r a_p, a_t \rangle\rangle = \langle\langle a_r, a_p a_t \rangle\rangle$  for all triples of points with  $r \sim p \perp t$  if and only if  $x^\perp$  is a subspace for all  $x \in \mathcal{P}$ .

We have

$$\begin{aligned}
\langle\langle a_r a_p, a_t \rangle\rangle - \langle\langle a_r, a_p a_t \rangle\rangle &= \frac{\eta}{2} \langle\langle a_r + a_p - a_s, a_t \rangle\rangle - \langle\langle a_r, 0 \rangle\rangle \\
&= \frac{\eta}{2} (\langle\langle a_r, a_t \rangle\rangle + \langle\langle a_p, a_t \rangle\rangle - \langle\langle a_s, a_t \rangle\rangle) \\
&= \frac{\eta}{2} (\langle\langle a_r, a_t \rangle\rangle - \langle\langle a_s, a_t \rangle\rangle) .
\end{aligned}$$

For this to be 0 we must have either  $r \perp t \perp s$  or  $r \sim t \sim s$ . As  $t \perp p$ , this says that  $t^\perp$  either contains all of the line  $\{p, r, s\}$  or it only contains  $p$ . This happens for all lines  $\{p, r, s\}$  with  $p \in t^\perp$  precisely when  $t^\perp$  is a subspace. This gives the claim.

*CLAIM. Assume that  $x^\perp$  is a subspace for all  $x \in \mathcal{P}$ . Then  $\langle\langle a_r a_p, a_t \rangle\rangle = \langle\langle a_r, a_p a_t \rangle\rangle$  for all triples of points with  $r \sim p \sim t$  if and only if, for all  $\{x, y, z\}$  and  $\ell = \{x, v, w\}$  lines of  $\mathcal{L}$ , we have  $y^\perp \cap \ell = \emptyset$  if and only if  $z^\perp \cap \ell = \emptyset$ .*

Let  $\{p, t, u\}$  be a line. As  $\langle\langle a_p, a_t \rangle\rangle = \frac{\eta}{2} = \langle\langle a_r, a_p \rangle\rangle$ ,

$$\begin{aligned}
\langle\langle a_r a_p, a_t \rangle\rangle - \langle\langle a_r, a_p a_t \rangle\rangle &= \frac{\eta}{2} (\langle\langle a_r + a_p - a_s, a_t \rangle\rangle - \langle\langle a_r, a_p + a_t - a_u \rangle\rangle) \\
&= \frac{\eta}{2} (\langle\langle a_r, a_t \rangle\rangle + \langle\langle a_p, a_t \rangle\rangle - \langle\langle a_s, a_t \rangle\rangle \\
&\quad - \langle\langle a_r, a_p \rangle\rangle - \langle\langle a_r, a_t \rangle\rangle + \langle\langle a_r, a_u \rangle\rangle) \\
&= \frac{\eta}{2} (-\langle\langle a_s, a_t \rangle\rangle + \langle\langle a_r, a_u \rangle\rangle) .
\end{aligned}$$

This is 0 when  $\langle\langle a_r, a_u \rangle\rangle = \langle\langle a_s, a_t \rangle\rangle$ ; that is, when we have

$$(*) \quad \text{either } r \sim u \text{ and } s \sim t \quad \text{or } r \perp u \text{ and } s \perp t .$$

We also have  $r \sim p \sim u$ , so we may replace  $t$  by  $u$  in the above to find that for the form to be associative on these two lines we must additionally have

$$(**) \quad \text{either } r \sim t \text{ and } s \sim u \quad \text{or } r \perp t \text{ and } s \perp u .$$

Conversely, the validity of  $(*)$  and  $(**)$  is sufficient for the form to be associative on the two lines.

As all  $q^\perp$  for  $q \in \{r, s, t, u\}$  are subspaces, each of  $\{r, s\}$  must be collinear with at least one of  $\{t, u\}$  and vice versa. This is equivalent to  $(*)$  and  $(**)$  except for the possibility that one of  $r$  and  $s$  is collinear with both of  $t$  and  $u$  while the other is collinear with only one. To avoid this, we want  $r$  to be collinear with both  $t$  and  $u$  if and only if  $s$  is as well. This is equivalent to requiring that  $r^\perp \cap \{p, t, u\}$  is empty if and only if  $s^\perp \cap \{p, t, u\}$  is empty.

This completes our proof of the second claim and so of the theorem.

□

**(7.3).** COROLLARY. *If  $\Pi$  is a Steiner triple system, then  $M(\Pi, \frac{\eta}{2}, \mathbb{F})$  admits an associative form, which is uniquely determined up to a scalar multiple as the form given by, for distinct  $p, q \in \mathcal{P}$ ,*

$$\langle\langle a_p, a_p \rangle\rangle = 1; \quad \langle\langle a_p, a_q \rangle\rangle = 0 \text{ if } q \in p^\perp; \quad \langle\langle a_p, a_q \rangle\rangle = \frac{\eta}{2} \text{ if } q \notin p^\perp.$$

PROOF. In Steiner triple systems, each  $x^\perp$  is empty.  $\square$

**(7.4).** COROLLARY. *If  $\Pi$  is a Fischer space, then  $M(\Pi, \frac{\eta}{2}, \mathbb{F})$  is a Frobenius axial algebra of Jordan type  $\eta$ . When  $\Pi$  is connected, an associative form is uniquely determined up to a scalar multiple as the form given by, for distinct  $p, q \in \mathcal{P}$ ,*

$$\langle\langle a_p, a_p \rangle\rangle = 1; \quad \langle\langle a_p, a_q \rangle\rangle = 0 \text{ if } q \in p^\perp; \quad \langle\langle a_p, a_q \rangle\rangle = \frac{\eta}{2} \text{ if } q \notin p^\perp.$$

PROOF. The fixed point subspace of the transposition  $t(x)$  is  $\{x\} \cup x^\perp$ , the disjoint union of  $\{x\}$  and the subspace  $x^\perp$ , giving (i) of Theorem (7.2). Furthermore, if  $\{x, y, z\}$  and  $\ell$  are two lines on the point  $x$ , then  $t(x)$  switches  $y$  and  $z$  and leaves  $\ell$  fixed globally, so that  $(y^\perp \cap \ell)^{t(x)} = z^\perp \cap \ell$ , giving (ii) of the theorem.  $\square$

This corollary has at least three uses. Especially, it completes our proof of Theorem (1.3).

By Proposition (2.7) and Theorem (6.2) every ideal of the algebra is a sum of ideals corresponding to connected components of  $\Pi$  and ideals contained in the radical, which is the maximal ideal containing no axes.

Finally, in the traditional applications the algebra is defined over  $\mathbb{R}$  and comes equipped with an associative, positive definite form. The form  $\langle\langle \cdot, \cdot \rangle\rangle$  of Corollary (7.4) has Gram matrix  $I + \frac{\eta}{2}D$ , where  $D$  is the adjacency matrix of the  $\Pi$ -collinearity graph on  $\mathcal{P}$ . In particular the form is positive (semi)definite when  $D$  has minimal eigenvalue greater than (or equal to)  $-2\eta^{-1}$ . For Griess and Majorana algebras, the cases of interest are  $\eta = \frac{1}{4}$  and  $\eta = \frac{1}{32}$ . These give the minimal eigenvalues  $-8$  (studied by Matsuo [Ma05]) and  $-64$  (considered by Hall and Shpectorov [HaSh]).

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